Problems with Variable Hilbert Space in Quantum Mechanics. Questions for Cosmology

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Basic. Problem for start of QM study

The infinite potential well U_i (i – *initial*) between points 0 and b

$$i\hbar \frac{d\psi(x,t)}{dt} = \hat{H}\psi(x,t)\,,$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + U_i(x)\,, \qquad U_i(x) = \begin{cases} \infty: & x \in \quad (-\infty, 0)\,, \\ 0: & x \in \quad [0, b]\,, \\ \infty: & x \in \quad (b, \infty)\,. \end{cases}$$

The stationary states of this problem are

$$|n\rangle_i \equiv |n; (0,b)\rangle \to \psi_{n,i} = \sqrt{\frac{2}{b}}\sin\frac{\pi n x}{b}\,, \qquad E_n = \frac{(\pi\hbar n)^2}{2mb^2}, \qquad n = 1,\, 2,\, \ldots$$

They form the basis for the Hilbert space $\mathcal{G}_i \equiv \mathcal{G}(0, b)$ of the continuous square-integrable functions on the interval [0, b], vanishing at its endpoints.

The stationary states for another (*ginal*) well f with boundaries (0, $b\alpha$) are described by similar equations with the change $b \to b\alpha$, e.g. $|n\rangle_f \equiv |n; (0, b\alpha)\rangle$. They form basis for another Hilbert space $\mathcal{G}_f \equiv \mathcal{G}(0, b\alpha)$.

Simple student problem

Let the width of the well changes instantly, $b \to b\alpha$.
To find probability W_{nk}^{if} of transition $|n\rangle_i \to |k\rangle_f$.

The result for the shrinking well ($\alpha < 1$):

$$W_{nk}^{if} \equiv \left|M_{nk}^{if}\right|^2 , \ \ where \ \ |M_{nk}^{if}| = \int_0^{b\alpha} \psi_{n,i}^*(x)\psi_{k,f}(x)dx$$

$$= \frac{2k\sqrt{\alpha}}{\pi} \cdot \left|\frac{\sin(\pi n\alpha)}{k^2 - (n\alpha)^2}\right| .$$

PROBLEM

A «simple» modification of the problem looks very natural:

> To find the same probability W_{nk}^{if} if the width changes as $b \to b\alpha(t)$ in a finite time T, with $\alpha(T) = \alpha$.

It looks natural to solve this problem with the aid of the Schrödinger equation with adding to the old potential in the perturbation

$$\hat{U}_P(x,t) = \begin{cases} 0: & x \in [0, b\alpha(t)]\,, \\ \infty: & x \in (b\alpha(t), b]; \end{cases} \quad at \ \ \alpha < 1\,,$$

$$\begin{cases} 0: & x \in [0, b]\,, \\ -\infty: & x \in (b, b\alpha(t)]\,. \end{cases} \quad at \ \ \alpha > 1\,.$$

The phenomena with the time dependent potential are described often with the aid of the time dependent perturbation theory (TDPT).
In the considered problem this approach is not applicable, since TDPT uses decomposition of time dependent wave functions in eigenfunctions of an initial state. However,
these functions belong to different Hilbert spaces $\mathcal{G}(0, b)$ and $\mathcal{G}(0, b\alpha(t))$, and, for example, at $\alpha(t) > 1$ these decomposition does not exist.

This situation generates two problems.
1. To present regular method for calculation of transfer probability, which allow to use some form of perturbation theory (or some other approximate method) in the cases when it looks natural.
2. How to characterize new phenomena (if they exist), appeared at the change of Hilbert space.

Regularization

The regularized problem is closer to the reality:
The infinite height walls are replaced by the walls of big height V, which does not changed when the well size is changed, i. e.

$$U_{reg}^{(1)}(x) = \begin{cases} V: & x \in (-\infty, 0)\,, \\ 0: & x \in (0, b)\,, \\ V: & x \in (b, \infty)\,, \end{cases} \qquad \left(V \gg \frac{(\pi\hbar)^2}{2mb^2}\right).$$

In this approach the change of potential is described by a simple substitution $b \to b\alpha(t)$. Both for initial and final states we deals with well known Hilbert space L_2 with $L_2 \supset \mathcal{G}_i, \mathcal{G}_f$.
Our problem corresponds to the limit $V \to \infty$ (removal of regularization).
With this regularization calculations become consistent but, unfortunately, very bulky.

Example of recipe for calculation

To solve the Schrödinger equation in our case we modify this equation by simple mapping of space, with introduction of new variable (rescaling)

$$\boxed{y = x/\alpha(t)\,.}$$

For this new variable the boundaries of well are fixed. The potential $U(y)$ keeps initial form U_i during the process. However, form of the kinetic term is changed strong $(d/dx \to \alpha^{-1}\partial/\partial y + y^{-1}\partial/\partial\alpha)$.
Besides, it is useful (but not obligatory) to modify scale of time and wave function in the following form

$$\tau = \int \frac{dt}{\alpha^2(t)}\,, \qquad \psi = \chi\sqrt{y}\,.$$

Calculations become transparent in the example with $\alpha(t) = 1 + [(\alpha - 1)t/T]$ (at $0 < t < T$). In this case

$$\tau = \frac{1}{\alpha'}\left(1 - \frac{1}{\alpha(t)}\right)\,, \quad \tau(t=0) = 0\,,\; \tau(t=T) = \frac{T}{\alpha}\,.$$

Now the Schrödinger equation is transformed to the form

$$i\hbar\frac{d\chi}{d\tau} = (\widehat{H}_0 + \widehat{U}_P)\chi\,, \quad \widehat{H}_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + U(y)\,,\ \widehat{U}_P = -\frac{\hbar^2}{2m}\hat{\tilde{\tilde{V}}}\,,$$
$$\hat{\tilde{\tilde{V}}} = \frac{2(1-\alpha'\tau)}{y\alpha'}\frac{\partial^2}{\partial y \partial \tau} + \left(\frac{1-\alpha'\tau}{y\alpha'}\right)^2 \cdot \frac{\partial^2}{\partial \tau^2} - \frac{3}{4y^2}.$$

$\widehat{H}_0$ is initial Hamiltonian.

Operator $\widehat{U}_P(y,\tau)$ is perturbation. In the case of fast but not very strong changes of well, $T \ll (\hbar(E_n - E_k))$ and $\alpha \sim 1$ transition probabilities $(n_i \to k_f)$ are calculable with the aid of standard time dependent perturbation theory.

The factors $1/y^2$ and $1/y$ in some terms of $\widehat{V}$ don't violate convergence of matrix elements V_{nk} due to boundary condition $\chi(0) = 0$.

Instant change of well. Shrinking well. Transitions to continuous spectrum

The probability for the transition of some state of incident well $|n\rangle_i$ into the any discrete state of final well $|k\rangle_f$ is

$$W(n;i|f) = \sum_k \left| M^{if}_{nk} \right|^2 = \langle n|_i \left\{ \sum_k |k\rangle_f \langle k|_f \right\} |n\rangle\_i \equiv \langle n|_i \mathbb{I}_f |n\rangle_i \,.$$

Here we define operator $\mathbb{I}_f \equiv \sum\limits_k |k\rangle_f \langle k|_f$. At the space $\mathcal{G}_f$ it acts as unit operator. This equation determine in fact how this operator acts in the other space $\mathcal{G}_i$ for our problem. It can be understood by two ways, giving coinciding results. First of all, one can summarize probabilities of individual transitions. Second, we use definition of operator $\mathbb{I}_f$ as the projector to the segment $(0, b\alpha)$.

- For the expanding well ($\alpha > 1$) we have

$$W(n;i|f) = \int_0^b dx \psi^*_{n,i}(x)\psi_{n,i}(x) = 1$$

(normalization). In other words, function $|n\rangle_i$, normalized on incident interval, keeps normalization after expansion of well.

• For the shrinking well ($\alpha < 1$) incident normalization integral lost interval $(b\alpha\,, b)$, so that we have $W(n;i|f) = \int\limits_0^{b\alpha} dx \psi^*_{n,i}(x)\psi_{n,i}(x) < 1$.

This probability is calculated by direct integration of basic solution and also by summation of series, obtained for separate transitions. The results naturally coincide

$$W(n;i|f) = \left\{ \begin{array}{l} \int\limits_0^{b\alpha} dx \dfrac{2}{b} \sin^2\left(\dfrac{\pi n x}{b}\right) \\ \dfrac{4\alpha}{\pi^2} \sum\limits_{k=1}^{\infty} \dfrac{k^2 \sin^2(\pi\beta)}{(k^2 - n\alpha^2)^2} \end{array} \right\} = \alpha\left(1 - \frac{\sin(2\pi n\alpha)}{2\pi n\alpha}\right) < 1\,.$$

Therefore at $\alpha < 1$ some fraction of probability disappear. In the above regularization picture it means that it some part of initial state goes over into the continuous spectrum, despite the fact that this spectrum disappears when the regularization is removed.

This phenomenon takes place also in the more general case of moving boundaries if final well don't cover initial one, for example at the shift of boundaries $(0\,,b) \to (a\,,c)$ with $a > 0$, $c > b$. (At $a > b$ transitions to the discrete spectrum are absent.)

Speculation. Comment for cosmology

We observe that the transition with the change of Hilbert space can be accompanied by departure of states into the unobservable, non-physical region.

The phase transition with breaking of Electroweak symmetry (after inflation) is accompanied by catastrophic change of the Hilbert space (at least, in perturbative approach).

• Before transition all particles were massless, Hilbert space was non-separable.

• After transition particles acquire masses, Hilbert space become separable (except photons). In addition, many particles, entered in the incident set of states, should be removed from this set, since they are unstable and, therefore, have no asymptotic states. In particular, W-bosons and muons are unstable, they should be removed from the basis of Hilbert space.

This rearrangement of Hilbert space can be accompanied lost of states.

If it is the fact, what is the fate of these lost states? Whether they can be source for dark energy

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