Introduction to neutrino mass models

Lecture 3: TBM from A_4 symmetry

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PMNS matrix

Quark masses in SM: single generation

Yukawa interactions provide masses to quarks:

$$
\begin{aligned}
-\mathcal{L}_Y^{(d)} &= y_d(\bar{Q}_L\Phi d_R + \bar{d}_R\Phi^\dagger Q_L) \to y_d(\bar{u}_L,\, \bar{d}_L)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} d_R + h.c. \\
&= \frac{y_d v}{\sqrt{2}}(\bar{d}_L d_R + \bar{d}_R d_L) \equiv m_d \bar{d} d\,.
\end{aligned}
$$

$$
-\mathcal{L}_Y^{(u)} = y_u(\bar{Q}_L\tilde{\Phi} u_R + \bar{u}_R\tilde{\Phi}^\dagger Q_L) \to y_u(\bar{u}_L,\, \bar{d}_L)\begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix} u_R + h.c. \equiv m_u \bar{u} u\,.
$$

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Quark masses and mixing

Three generations Q_{Li}, d_{Ri}, u_{Ri}, $i = 1, 2, 3$:

$$d_i = (d,\, s,\, b) \quad u_i = (u,\, c,\, t)\,.$$

Yukawa interactions are parametrized with coupling matrices Γ_{ij} and Δ_{ij}:

$$\begin{aligned} -\mathcal{L}_Y &= \bar{Q}_{Li}\Gamma_{ij}\Phi d_{Rj} + \bar{Q}_{Li}\Delta_{ij}\tilde{\Phi} u_{Rj} + h.c. \\ &\to \bar{d}_{Li}(M_d)_{ij} d_{Rj} + \bar{u}_{Li}(M_u)_{ij} u_{Rj} + h.c. \end{aligned}$$

where the 3×3 mass matrices are

$$(M_d)_{ij} = \Gamma_{ij}\frac{v}{\sqrt{2}}\,, \quad (M_u)_{ij} = \Delta_{ij}\frac{v}{\sqrt{2}}$$

and are, in general, non-diagonal and complex.

CKM matrix

M_d is diagonalized by $d_L = V_{dL} d_L^{phys}$, $d_R = V_{dR} d_R^{phys}$, and so is M_u:

$$V_{dL}^\dagger \, M_d \, V_{dR} = D_d = \mathrm{diag}(m_d, \, m_s, \, m_b) \,,$$

$$V_{uL}^\dagger \, M_u \, V_{uR} = D_u = \mathrm{diag}(m_u, \, m_c, \, m_t) \,,$$

But then the charged current matrix can become non-trivial:

$$\bar{u}_{Li} \, \gamma^\mu W_\mu^+ \, d_{Li} = \bar{u}_{Li}^{phys} \, \gamma^\mu W_\mu^+ \, V_{ij} d_{Lj}^{phys} \,, \quad \text{where} \quad V_{ij} = V_{uL}^\dagger V_{dL} \neq \delta_{ij} \,.$$

Conclusion

if coupling matrices Γ_{ij} and Δ_{ij} are distinct,
then quark mass eigenstates $\neq$ charged current eigenstates.

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Lepton mixing: Dirac

Massive neutrinos implies that they are either Dirac or Majorana.

For Dirac neutrinos, we add ν_{Ri}, $i = 1, 2, 3$, write only Dirac mass term, get lepton mass matrices M_ℓ and $\mathcal{M}_\nu$, and diagonalize them as before:

$$U^\dagger_{\ell L}\, M_\ell\, U_{\ell R} = D_\ell = \mathrm{diag}(m_e,\, m_\mu,\, m_\tau)\,,$$
$$U^\dagger_{\nu L}\, \mathcal{M}_\nu\, U_{\nu R} = D_\nu = \mathrm{diag}(m_1,\, m_2,\, m_3)\,,$$

The charged weak currents are written in the generation space as

$$\underbrace{\overline{\ell_{Li}}\gamma^\mu W^-_\mu \nu_{Li}}_{\text{original}} = \underbrace{(\overline{e_L}, \overline{\mu_L}, \overline{\tau_L})\gamma^\mu W^-_\mu \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}}_{\text{flavor basis}} \equiv (\overline{e_L}, \overline{\mu_L}, \overline{\tau_L})\gamma^\mu W^-_\mu\, \boxed{U_{PNMS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}}$$

Lepton mixing: Dirac

$$\underbrace{\overline{\ell_{Li}}\gamma^\mu W_\mu^- \nu_{Li}}_{\text{original}} = \underbrace{(\overline{e_L}, \overline{\mu_L}, \overline{\tau_L})\gamma^\mu W_\mu^- \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}}_{\text{flavor basis}} \equiv (\overline{e_L}, \overline{\mu_L}, \overline{\tau_L})\gamma^\mu W_\mu^- \, U_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Flavor basis is defined as the charged lepton mass basis:

$$\ell_{Li} = U_\ell \ell_L^{\text{mass}}\,, \quad \nu_{Li} = U_\nu \nu_L^{\text{mass}}$$

Therefore, the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix is

$$\boxed{U_{PMNS} = U_\ell^\dagger U_\nu \,.}$$

If M_ℓ is already diagonal, then $U_{PMNS} = U_\nu$.

Lepton mixing: Dirac

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

After removing phases, the standard parametrization is

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

Since $\mathcal{M}_\nu$ is diagonalized by bi-unitary transformation

$$U^\dagger_{\nu L}\,\mathcal{M}_\nu\,U_{\nu R} = D_\nu = \mathrm{diag}(m_1,\, m_2,\, m_3)\,,$$

some phases from $U_{\nu L}$ can be moved to $U_{\nu R}$.

PMNS matrix U_{PMNS} contains only one irremovable phase.

Lepton mixing: Dirac

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Lepton mixing: Majorana

For Majorana neutrinos, the mass matrix is

$$\nu_{Li}^T (\mathcal{M}_\nu)_{ij} \mathcal{C} \nu_{Lj} = (\nu_L^{\text{mass}})^T \, U_\nu^T \mathcal{M}_\nu U_\nu \mathcal{C} \nu_L^{\text{mass}} = (\nu_L^{\text{mass}})^T D_\nu \mathcal{C} \nu_L^{\text{mass}}$$

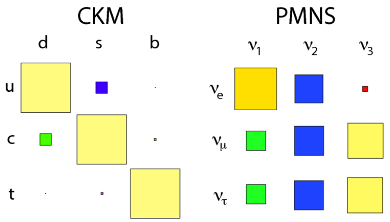
with the same matrix U_ν on both sides.

One can always find such U_ν to make D_ν diagonal with real posivite values. But once this is done, there is no freedom left to remove phases!

$$U_{PMNS}^{\text{Majorana}} = U_{PMNS} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} .$$

These two additional Majorana phases are the echo of the complex neutrino mass matrix $\mathcal{M}_\nu$.

Lepton mixing



U_{PMNS} is close to the tri-bimaximal mixing pattern [Harrison, Perkins, Scott, 2002]:

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Nonzero s_{13} highlights deviation, but proximity of U_{PMNS} to the TBM is indicative of some symmetry.

Basics of finite group theory

Groups

Set G is a group if it satisfies the following four axioms:

- closure of G under composition (usually called multiplication):

$$\text{for any } g_1,\, g_2 \in G,\ \text{ define their product } g_1 \cdot g_2 \in G\,;$$

- the multiplication is associative: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for all $g_1, g_2, g_3 \in G$;
- there exists a special element called identity element e with the properties:

$$g \cdot e = e \cdot g = g \qquad \text{for any } g \in G;$$

- every element is invertible: for any $g \in G$, there exists another element in G (denoted g^{-1}) such that

$$g^{-1} \cdot g = g \cdot g^{-1} = e\,.$$

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Groups

In addition, if $g \cdot h = h \cdot g$ for all elements $g, h \in G$, the group is called abelian. If it fails at least for one pair, the group is called non-abelian.

Non-abelian groups are much, much, much more complicated than abelian groups.

Groups arise in physics in the context of transformations and symmetries. It is the most appropriate language to describe hidden consequences of physics formulas or laws.

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Groups

Groups can be finite or infinite.

- A finite group G has finite number of elements: $G = \{e, g_2, g_3, \dots, g_n\}$. Its size n is called the order of the group and is denoted $|G|$.
- In a finite group, successive multiplications will sooner or later terminate in e. Pick up any $g \in G$ and consider successive powers:

$$g^1 \equiv g \quad g^2 \equiv g \cdot g\,, \quad g^3 \equiv g \cdot g \cdot g\,, \quad g^k \equiv \underbrace{g \cdot \cdots \cdot g}_{k \text{ times}}\,.$$

  Then, there must exist an integer p such that $g^p = e$. This integer p is called the order of the element g.
- Infinite groups can be discrete or continuous (= topological).

Basics examples

- Integers $\mathbb{Z}$ and reals $\mathbb{R}$ are groups under addition. The identity element is 0. They are not groups under multiplication!

- Reals on the interval [0, 1] form a group under addition with periodic boundary condition ($0.999\cdots = 0$). These are fractional part of reals: $\mathbb{R}/\mathbb{Z}$.

  Complex numbers with $|z| = 1$ form under multiplication the circle group, or the rephasing group $U(1)$.

  The two last groups are isomorphic: $\mathbb{R}/\mathbb{Z} \simeq U(1)$.

- Cyclic groups $\mathbb{Z}_n$ for any $n > 1$ are defined as

$$\mathbb{Z}_n = \{e,\, a,\, a^2,\, a^3,\, \dots,\, a^{n-1}\} \quad \text{with condition} \quad a^n = e\,,$$

  isomorphic to integers modulo n under addition: $\mathbb{Z}/n\mathbb{Z}$.

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Presentation of a group

How would you describe a finite group?

Simplest choice: write the multiplication table $|G| \times |G|$. Very impractical.

Much better choice: via generators and relations.

- Generators $a, b, c, \ldots$ form a subset of elements of G such that any $g \in G$ can be written and their product.
- Generators are independent elements but they satisfy some constraints (relations).
- Group presentation: $G = \langle\, \text{generators} \,|\, \text{their relations} \,\rangle$.
- A cyclic group is generated by a: $\mathbb{Z}_n = \langle a \,|\, a^n = e \rangle$.
  Direct product of cyclic groups: $\mathbb{Z}_n \times \mathbb{Z}_m = \langle a,\, b \,|\, a^n = b^m = e,\, ab = ba \rangle$.

Representations of abelian groups

A representation of the group G is, colloquially speaking, a way of rewriting it as a group of matrices which act on some k-dimensional vector space.

The set of matrices must obey exactly the same rules as the elements of G, but otherwise there is no constraints on their form or dimension k. For example,

$$\mathbb{Z}_2 = (e, a): \qquad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbb{Z}_3 = (e, b, b^2): \qquad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

where $\omega \equiv \exp(2\pi i/3)$, $\omega^3 = 1$.

Representations of abelian groups

General theorem: for any abelian unitary group, the representing matrices can be always made diagonal by a basis choice.

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2 = (e, a, b, ab)$ with a faithfull 2D representation:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad ab = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this basis, each 1D subspace remains invariant; and the diagonal numbers form a 1D representation.

$$\text{subspace} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad a = 1,\ b = -1\,,$$

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Working example: A_4

Non-abelian groups

There is a much richer list of (finite) non-abelian groups. Some examples:

- S_n, group of all permutations of n elements. Its order is $|S_n| = n!$. The smallest group is $S_2 \simeq \mathbb{Z}_2$. The smallest non-abelian is

$$S_3 = \langle a, b | a^2 = b^3 = e, ab = b^2 a \rangle \,.$$

- A_n, group of even-signature permutations of n elements; $|A_n| = n!/2$.
- Symmetry groups of regular polygons and polyhedra:
  - Symmetry group of equilateral triangle $\simeq S_3$;
  - Symmetry group of tetrahedron $\simeq A_4$;
  - Symmetry group of cube $\simeq S_4$.

Irreducible representations of non-abelian groups have $d > 1$.

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Group A_4

A_4 is the smallest group with irreducible 3D representation:

$$A_4 = \langle S, T | S^2 = T^3 = e, (ST)^3 = e \rangle \,, \quad |A_4| = 12\,.$$

It contains:

- three elements of order 2: S, T^2ST, TST^2;
- together with e, they form the Klein subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$;
- four cycles of order 3 generated by T, ST, TS, T^2ST^2 (8 elements of order 3 in total).

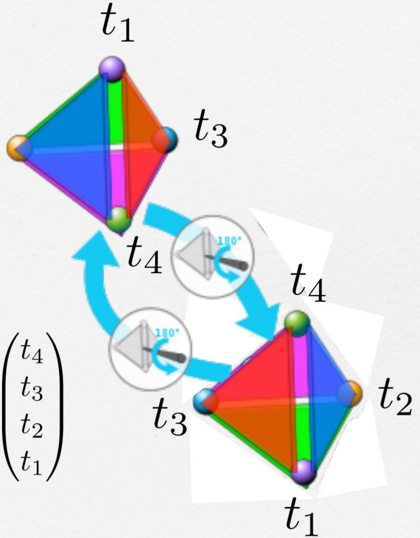
A_4: transformation S

A4

- rotation by 180°

$$S$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_4 \\ t_3 \\ t_2 \\ t_1 \end{pmatrix}$$



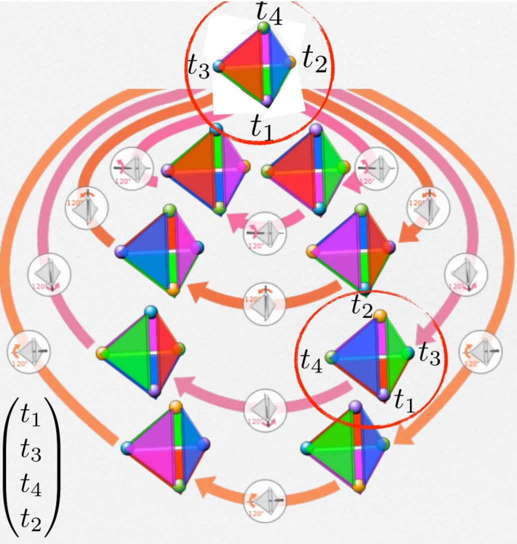
A_4: transformation T

A4

- rotation by 120° anti-clockwise (seen from a vertex)

$$T$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_3 \\ t_4 \\ t_2 \end{pmatrix}$$



Group A_4

3D irreducible representation: diagonal-S basis

- order 2:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T^2ST = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad TST^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- order 3:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad ST = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$TS = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T^2ST^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

and their squares.

Group A_4

3D irreducible representation: diagonal-T basis

One can switch to another basis in the same 3D space, in which T becomes diagonal.

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega \equiv e^{2\pi i/3}, \quad \omega^3 = 1\,.$$

Then, S takes an “ugly” shape:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Nevertheless, all group multiplications hold: $S^2 = e$, etc

Group A_4

Subspaces in the diagonal-T basis are convenient to define three non-equivalent 1D irreps: 1, $1'$, $1''$

The full table of all irreps of A_4:

irrep	S	T
1	$S = 1$	$T = 1$
$1'$	$S = 1$	$T = \omega$
$1''$	$S = 1$	$T = \omega^2$
3	matrix S	matrix T

Notice: the trivial singlet 1 is invariant under the entire A_4.

Building symmetry-based models

with the example of A_4 3HDM

Tensor product decomposition

Models begin with lagrangian $\mathcal{L}$, which encodes all interactions.

Terms in the lagrangian are products of various fields:

$$\mathcal{L} = \cdots + \lambda_5(\Phi_1^\dagger \Phi_2)^2 + \cdots + Y_{ij}^a \overline{Q_{Li}} \Phi_a d_{Rj} + \cdots$$

We assume that each set of fields (LH fermions, RH fermions, Higgses, etc) transforms as a certain representation of group G.

We want to find which combinations are fully G-invariant.

We must use the tensor product of representations.

Tensor product decomposition

Take 3D vectors $a_i = (a_1, a_2, a_3)$ and $b_j = (b_1, b_2, b_3)$ and construct their tensor product $a_i b_j$. How does it transform under $SO(3)$ rotations?

$$a_i b_j = \delta_{ij}\frac{(\vec{a}\vec{b})}{3} + \epsilon_{ijk}\cdot \underbrace{v_k}_{=[\vec{a}\times\vec{b}]/2} + \left[\frac{1}{2}(a_i b_j + a_j b_i) - \delta_{ij}\frac{(\vec{a}\vec{b})}{3}\right],$$

which means that inside the 9D tensor $a_i b_j$ there are three invariant subspaces: singlet, $\propto \delta_{ij}$; triplet, $\propto \epsilon_{ijk} v_k$, and 5-plet, the traceless symmetric part of $a_i b_j$.

Group-theoretically: $3 \otimes 3 = 1 \otimes 3 \otimes 5$.

This is how tensor product decomposition (= Clebsch-Gordan coefs) works in the group $SO(3)$.

Tensor product decomposition

For each group, these rules are different (= Clebsch-Gordan coefs are different).

For A_4, if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two irreducible triplets, then

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_1 \oplus 3_2 \,.$$

The explicit expressions for their components (in the S-symmetric basis!) are:

$$\begin{aligned}
1 &= a_1 b_1 + a_2 b_2 + a_3 b_3 \,, \\
1' &= a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \,, \\
1'' &= a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \,, \\
3_1 &= (a_2 b_3, a_3 b_1, a_1 b_2) \,, \\
3_2 &= (a_3 b_2, a_1 b_3, a_2 b_1) \,.
\end{aligned}$$

The products of singlets are intuitive: $1' \otimes 1'' = 1$, etc.

Picking up symmetric terms

When building symmetry-constrained lagrangians, we

- write products of fields, each transforming as a certain irrep of the group G,
- perform tensor product decomposition,
- out of all final irreps, keep only trivial singlets as they are G-symmetric.

For example, in three-Higgs-doublet model based on group A_4, we have three Higgs doublets Φ_1, Φ_2, Φ_3. In general, the quadratic part of the potential has nine terms $\Phi_i^\dagger \Phi_j$.

But knowing that, for the group A_4, $3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_1 \oplus 3_2$, we keep only the singlet. Therefore, the Higgs potential is

$$V = -m^2 \left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3\right) + V_4$$

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Picking up symmetric terms

For the quartic part, we decompose $(\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l)$,

$$[(3 \otimes 3) \otimes (3 \otimes 3)]_{sym} = [(1 \oplus 1' \oplus 1'' \oplus 3_1 \oplus 3_2) \otimes (1 \oplus 1' \oplus 1'' \oplus 3_1 \oplus 3_2)]_{sym}$$

$$= 1 \otimes 1 + 1' \otimes 1'' + \underbrace{(3_1 \otimes 3_1)}_{=1\oplus\ldots} + \underbrace{(3_2 \otimes 3_2)}_{=1\oplus\ldots} + \underbrace{(3_1 \otimes 3_2)}_{=1\oplus\ldots} + \ldots\,,$$

which gives five trivial singlets 1:

$$\begin{aligned} V_4 &= \lambda_1 \left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 + \Phi_3^\dagger \Phi_3\right)^2 \\ &+ \lambda_2 \left[(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + (\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3) + (\Phi_3^\dagger \Phi_3)(\Phi_1^\dagger \Phi_1)\right] \\ &+ \lambda_3 \left[(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + (\Phi_2^\dagger \Phi_3)(\Phi_3^\dagger \Phi_2) + (\Phi_3^\dagger \Phi_1)(\Phi_1^\dagger \Phi_3)\right] \\ &+ \left(\lambda_4 \left[(\Phi_1^\dagger \Phi_2)^2 + (\Phi_2^\dagger \Phi_3)^2 + (\Phi_3^\dagger \Phi_1)^2\right] + h.c.\right) \end{aligned}$$

Spontaneous symmetry breaking

In this way, we get the full A_4-symmetric potential in 3HDM.

But the minimum of this potential (v_1, v_2, v_3) may break this group, fully or completely. Which options are available for the minimum in the A_4-symmetric 3HDM?

It turns out that vevs (v_1, v_2, v_3) cannot be arbitrary! Depending on paremeters λ, only four vev alignments are possible [Degee, Ivanov, Keus, 2012]:

- $(1,0,0)$. The residual symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- $(1,1,1)$. The residual symmetry group is $\mathbb{Z}_3$.
- $(1,\omega,\omega^2)$. The residual symmetry group is $\mathbb{Z}_3$.
- $(1,e^{i\alpha},0)$. The residual symmetry group is $\mathbb{Z}_2$.

Conclusion: it is impossible to break the A_4 symmetry completely.

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Conclusion: it is impossible to break the A_4 symmetry completely.

Extending A_4 3HDM to charged leptons

Extending A_4 symmetry of 3HDM to the Majorana LH neutrino mass matrix [Gonzales Felipe, Serodio, Silva, 2013].

Charged lepton Yukawa interactions:

$$\overline{L_i} Y^a_{ij} \underbrace{\Phi_a}_{3} \ell_{Rj} + h.c.$$

We know that $\Phi_a = (\Phi_1, \Phi_2, \Phi_3)$ transforms as triplet 3 under A_4.

Therefore, the product of L_i and ℓ_{Rj} must also transform as a triplet 3 to produce the trivial singlet 1 at the end.

L_i	ℓ_{Rj}
3	3
(1, 1', 1")	3
3	(1, 1', 1")

Extending A_4 3HDM to charged leptons

For example, if $\overline{L_i} \sim (1, 1', 1'')$ and $\ell_{Rj} \sim 3$, we get:

$$\begin{aligned}
\overline{L_i} Y^a_{ij} \Phi_a \ell_{Rj} &= y_1 \overline{L_1} \underbrace{\Phi_a \ell_{Rj}}_{1} + y_2 \overline{L_2} \underbrace{\Phi_a \ell_{Rj}}_{1''} + y_3 \overline{L_3} \underbrace{\Phi_a \ell_{Rj}}_{1'} \\
&= y_1 \overline{L_1}(\Phi_1 \ell_{R1} + \Phi_2 \ell_{R2} + \Phi_3 \ell_{R3}) \\
&\quad + y_2 \overline{L_2}(\Phi_1 \ell_{R1} + \omega \Phi_2 \ell_{R2} + \omega^2 \Phi_3 \ell_{R3}) \\
&\quad + y_3 \overline{L_3}(\Phi_1 \ell_{R1} + \omega^2 \Phi_2 \ell_{R2} + \omega \Phi_3 \ell_{R3})
\end{aligned}$$

Pick up a vev alignment, for example, $v(1, 1, 1)$. Then, charged lepton mass matrix is

$$M_\ell = v \begin{pmatrix} y_1 & y_1 & y_1 \\ y_2 & \omega y_2 & \omega^2 y_2 \\ y_3 & \omega^2 y_3 & \omega y_3 \end{pmatrix},$$

which, after diagonalization gives $m_\ell = \{y_1 v, y_2 v, y_3 v\} \to$ OK.

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&= y_1 \overline{L_1} (\Phi_1 \ell_{R1} + \Phi_2 \ell_{R2} + \Phi_3 \ell_{R3}) \\
&\quad + y_2 \overline{L_2} (\Phi_1 \ell_{R1} + \omega \Phi_2 \ell_{R2} + \omega^2 \Phi_3 \ell_{R3}) \\
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\end{aligned}$$

Pick up a vev alignment, for example, $v(1,1,1)$. Then, charged lepton mass matrix is

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which, after diagonalization gives $m_\ell = \{y_1 v,\ y_2 v,\ y_3 v\} \to$ OK.

Extending A_4 3HDM to Majorana neutrinos

Then, include Majorana neutrino terms:

$$c_{ij}^{ab}(L_i^T\tilde{\Phi}_a^*)\,\mathcal{C}\,(\tilde{\Phi}_b^\dagger L_j)\,.$$

Group-theoretically, we see

$$(L\otimes L)\otimes(\underbrace{\tilde{\Phi}}_{3}\otimes\underbrace{\tilde{\Phi}}_{3})$$

Since $\overline{L_i}\sim(1,1',1'')$, the product $L\otimes L$ also contains 1, $1'$, and $1''$, which are coupled to $3\otimes 3$:

$$\begin{aligned}
&\frac{g_1}{\Lambda}(L_1L_1+L_2L_3+L_3L_2)(\tilde{\Phi}_1\tilde{\Phi}_1+\tilde{\Phi}_2\tilde{\Phi}_2+\tilde{\Phi}_3\tilde{\Phi}_3)\\
+\quad&\frac{g_2}{\Lambda}(L_1L_2+L_2L_1+L_3L_3)(\tilde{\Phi}_1\tilde{\Phi}_1+\omega\tilde{\Phi}_2\tilde{\Phi}_2+\omega^2\tilde{\Phi}_3\tilde{\Phi}_3)\\
+\quad&\frac{g_3}{\Lambda}(L_1L_3+L_2L_2+L_3L_1)(\tilde{\Phi}_1\tilde{\Phi}_1+\omega^2\tilde{\Phi}_2\tilde{\Phi}_2+\omega\tilde{\Phi}_3\tilde{\Phi}_3)
\end{aligned}$$

Extending A_4 3HDM to Majorana neutrinos

Next, substituting the chosen vev alignment $(1,1,1)$, we get neutrino mass matrix:

$$\mathcal{M}_\nu = \frac{g_1 v^2}{2\Lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

We obtain three degenerate neutrinos!

Conclusion

extending A_4 symmetry to charged leptons and Majorana neutrinos with irrep assignment

$$\Phi \sim 3\,, \quad L \sim (1, 1', 1'')\,, \quad \ell_R \sim 3$$

and with the vev alignment $\langle \phi^0 \rangle = v(1,1,1)$ is ruled out by experiment.

Extending A_4 3HDM to Majorana neutrinos

One needs to check all possible irrep assignments and all possible vev alignments. This was done in [Gonzales Felipe, Serodio, Silva, 2013].

The result is: all possible combinations are ruled out experimentally. The problems can be:

- massless charged leptons,
- degenerate neutrino masses,
- insufficient neutrino mixing.

Thus, 3HDM scalar sector offers too little freedom to produce viable Majorana neutrino masses through the A_4 symmetry group.

One needs to enlarge the scalar sector to get a viable neutrino sector.

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One needs to enlarge the scalar sector to get a viable neutrino sector.

TBM PMNS from A_4 symmetry

Classic seesaw again

Leptonic Yukawas:

$$
\begin{aligned}
& \overline{L_i} Y^{\ell}_{ij} \Phi \ell_{Rj} + \overline{L_i} Y^{\nu}_{ij} \tilde{\Phi} \nu_{Rj} + \frac{1}{2} \overline{(\nu_{Ri})^c} (M_R)_{ij} \nu_{Rj} + h.c. \\
= \quad & \overline{\ell_L} M_\ell \ell_R + \frac{1}{2} \left[\overline{\nu_L}, \overline{(\nu_R)^c} \right] \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + h.c.
\end{aligned}
$$

which leads to

$$\mathcal{M}_\nu = -m_D (M_R)^{-1} m_D^T \,.$$

The classic seesaw does not constrain matrices m_D and M_R $\to$ no predictions on $\mathcal{M}_\nu$ $\to$ no predictions on U_{PMNS}.

Flavons

Flavor symmetry-based modes assume that L_i, $(\ell_R)_i$, and $(\nu_R)_i$ transform in certain way under a discrete flavor symmetry group G.

Problem: combining L, ℓ_R, and ν_R, via only Higgs doublets leads to contradiction to experiment!

Recipe: leave the poor Φ alone! Add flavons φ_a: new auxiliary scalar fields, which will take care of the shape of the fermion sector.

Flavons

Flavons

- EW singlets → play no role in electroweak symmetry breaking;
- transform non-trivially under G:

$$\text{instead of} \quad Y_{ij}\overline{L_i}\Phi\ell_{Rj} \quad \text{we use} \quad Y_{ij}^a \frac{\varphi_a}{\Lambda}(\overline{L_i}\Phi\ell_{Rj})\,.$$

  Symmetry under G strongly constraints Y_{ij}^a.
- they get vev after minimization of flavon scalar potential $\varphi_a \to \langle\varphi_a\rangle \to$ spontaneous breaking of flavor symmetry induces usual Yukawa interactions

$$Y_{ij} = \sum_a Y_{ij}^a \frac{\langle\varphi_a\rangle}{\Lambda}\,.$$

  with the resulting Y_{ij} constrained by flavor symmetry.

A_4 symmetric model

	$\bar{L}$	e_R	μ_R	τ_R	ν_R	Φ	φ_T	φ_S	ξ
$SU(2)_L$	$\bar{2}$	1	1	1	1	2	1	1	1
A_4	3	1	$1'$	$1''$	3	1	3	3	1

We assume:

- LH doublets L and RH neutrinos ν_R form A_4 triplets;
- right charged leptons e_R, μ_R, τ_R form three different A_4 singlets $1, 1', 1''$;
- add three sorts of flavons:
  - A_4-triplet φ_T helps join L with ℓ_R ($\to$ charged lepton masses),
  - A_4-triplet φ_S produces one Majorana term for ν_R,
  - A_4-singlet ξ produces another Majorana term for ν_R.

A_4 symmetric model

Warning: this construction is not yet self-consistent! So far nothing explains why φ_T and φ_S play different roles!

To forbid "wrong terms", we introduce yet another quantum number: "charge" under the group $\mathbb{Z}_3$.

	$\bar{L}$	e_R	μ_R	τ_R	ν_R	Φ	φ_T	φ_S	ξ
$SU(2)_L$	$\bar{2}$	1	1	1	1	2	1	1	1
A_4	3	1	$1'$	$1''$	3	1	3	3	1
$\mathbb{Z}_3$	ω^2	ω	ω	ω	ω	1	1	ω	ω

The true symmetry group of the model is $A_4 \times \mathbb{Z}_3$ but I will skip $\mathbb{Z}_3$ for clarity.

A_4 symmetric model

TMB mixing from A_4 symmetry [Altarelli, Feruglio, 2006]

	$\bar{L}$	e_R	μ_R	τ_R	ν_R	Φ	φ_T	φ_S	ξ
$SU(2)_L$	$\bar{2}$	1	1	1	1	2	1	1	1
A_4	3	1	$1'$	$1''$	3	1	3	3	1

$$
\begin{aligned}
\mathcal{L} \quad &= \quad \text{charged leptons separately for } e,\ \mu, \text{ and } \tau \\
&\qquad + \text{Dirac mass term} + \text{two Majorana mass terms} \\
&= \quad \frac{y_e}{\Lambda}\underbrace{(\bar{L}\varphi_T)}_{3\times3\to1}\underbrace{e_R}_{1}\Phi + \frac{y_\mu}{\Lambda}\underbrace{(\bar{L}\varphi_T)}_{3\times3\to1''}\underbrace{\mu_R}_{1'}\Phi + \frac{y_\tau}{\Lambda}\underbrace{(\bar{L}\varphi_T)}_{3\times3\to1'}\underbrace{\tau_R}_{1''}\Phi \\
&\qquad + y_D\underbrace{(\bar{L}\nu_R)}_{3\times3\to1}\tilde{\Phi} + y_a\underbrace{(\nu_R\nu_R)\xi}_{(3\times3)_1\times1} + y_b\underbrace{(\nu_R\nu_R\varphi_S)}_{3\times3\times3\to1} + h.c.
\end{aligned}
$$

A_4 symmetric model

It is convenient to work in the T-diagonal basis → charged lepton mass matrix will be diagonal.

Contractions written explicitly (for explicit expressions, see e.g. [Altarelli, Feruglio, 1002.0211]):

$$\begin{aligned}
(\overline{\ell_L}\varphi_T)_1 &= \overline{e_L}(\varphi_T)_1 + \overline{\mu_L}(\varphi_T)_3 + \overline{\tau_L}(\varphi_T)_2 \\
(\overline{\ell_L}\varphi_T)_{1''} &= \overline{e_L}(\varphi_T)_2 + \overline{\mu_L}(\varphi_T)_1 + \overline{\tau_L}(\varphi_T)_3 \\
(\overline{\ell_L}\varphi_T)_{1'} &= \overline{e_L}(\varphi_T)_3 + \overline{\mu_L}(\varphi_T)_2 + \overline{\tau_L}(\varphi_T)_1
\end{aligned}$$

and $(\nu_R\nu_R\varphi_S)_1$ gives

$$\nu_{Ri}\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\varphi_{S1} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}\varphi_{S2} + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\varphi_{S3}\right]\nu_{Rj}$$

Flavons alignment

The flavon potential is also A_4 symmetric and has "Mexican hat" form. It produces nonzero vevs with the following alignment:

$$\langle\varphi_T\rangle \propto (1,0,0)\,, \quad \langle\varphi_S\rangle \propto (1,1,1)\,, \quad \langle\xi\rangle \neq 0\,.$$

This is the vev alignment which we postulate when building our model.

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\,, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}\,, \quad \omega \equiv e^{2\pi i/3}\,.$$

- $\langle\varphi_T\rangle$ conserves T (subgroup $\mathbb{Z}_3$),
- $\langle\varphi_S\rangle$ conserves S (and TST^2, subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$).

$$\begin{tikzcd} & A_4 \arrow[dl, "\langle\varphi_T\rangle"'] \arrow[dr, "\langle\varphi_S\rangle"] & \\ \mathbb{Z}_3 & & \mathbb{Z}_2 \times \mathbb{Z}_2 \end{tikzcd}$$

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Consequences of flavons alignment

Charged leptons $\langle\varphi_T\rangle \sim (1,0,0)$:

$$\frac{y_e}{\Lambda}(\bar{L}\varphi_T)e_R\Phi + \frac{y_\mu}{\Lambda}(\bar{L}\varphi_T)\mu_R\Phi + \frac{y_\tau}{\Lambda}(\bar{L}\varphi_T)\tau_R\Phi$$

$$\to \quad \frac{\langle\varphi_T\rangle}{\Lambda}\left(y_e\overline{L_e}e_R + y_\mu\overline{L_\mu}\mu_R + y_\tau\overline{L_\tau}\tau_R\right)\Phi + h.c.$$

$$\to \quad (\overline{e_L},\overline{\mu_L},\overline{\tau_L})\begin{pmatrix} m_e & 0 & 0\\ 0 & m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}\begin{pmatrix} e_R\\ \mu_R\\ \tau_R\end{pmatrix} + h.c.$$

Charged lepton mass matrix is automatically diagonal.

Consequences of flavons alignment

Dirac mass matrix:

$$y_D(\bar{L}\nu_R)\tilde{\Phi} = y_D\overline{\nu_{Li}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\nu_{Rj}\tilde{\Phi}\,,$$

which after EWSB gives $\overline{\nu_{Li}}(m_D)_{ij}\nu_{Rj}$ with

$$m_D = \frac{y_D v}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Consequences of flavons alignment

Majorana mass matrix for ν_R:

$$M_R = y_a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \langle\xi\rangle$$

$$+ y_b \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \langle\varphi_{S1}\rangle + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \langle\varphi_{S2}\rangle + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \langle\varphi_{S3}\rangle \right]$$

$$= a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} .$$

Consequences of flavons alignment

Overall result:

$$\mathcal{M}_\nu = -m_D (M_R)^{-1} m_D^T \,,$$

where

$$m_D = \frac{y_D v}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,, \quad M_R = \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix} \,.$$

Then we will need to diagonalize it:

$$D_\nu = U^T \mathcal{M}_\nu U \,,$$

and, since the charged lepton matrix is already diagonal,

$$U_{PMNS} = U \,.$$

Inverting M_R

The simple form of

$$M_R = \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix}.$$

allows to explicitly calculate eigenvalues and eigenvectors:

$$\lambda = 3b + a\,, \qquad \vec{v} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix},$$

$$\lambda = a\,, \qquad \vec{v} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda = 3b - a\,, \qquad \vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Inverting M_R

This means that

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 3b+a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 3b-a \end{pmatrix}$$

Or, in short,

$$U^T M_R U = D_R \quad \Leftrightarrow \quad U D_R U^T = M_R\,.$$

Calculating $\mathcal{M}_\nu$

Therefore,

$$M_R^{-1} = UD_R^{-1}U^T = U \begin{pmatrix} \frac{1}{3b+a} & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{3b-a} \end{pmatrix} U^T .$$

The light neutrino mass matrix is therefore

$$\begin{aligned} \mathcal{M}_\nu &= m_D M_R^{-1} m_D^T = m_D \cdot UD_R^{-1}U^T \cdot m_D^T \\ &= U \cdot U^T m_D U \cdot D_R^{-1} \cdot U^T m_D^T U \cdot U^T . \end{aligned}$$

We need to calculate $U^T m_D U$.

Calculating $\mathcal{M}_\nu$

Notice that

$$m_D = \frac{y_D v}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

is exactly like the a-term of M_R. Therefore,

$$U^T m_D U = \frac{y_D v}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Overall result:

$$\mathcal{M}_\nu = U(U^T m_D U)\, D_R^{-1}\, (U^T m_D^T U)\, U^T = \frac{y_D^2 v^2}{2} U\, D_R^{-1}\, U^T.$$

Consequences

$$\mathcal{M}_\nu = \frac{y_D^2 v^2}{2} U \begin{pmatrix} \frac{1}{3b+a} & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{3b-a} \end{pmatrix} U^T .$$

- U diagonalizes $\mathcal{M}_\nu \to U = U_{PMNS}$ is of the TBM form;
- neutrino masses are:
$$m_1 = \frac{y_D^2 v^2}{3b+a}, \quad m_2 = \frac{y_D^2 v^2}{a}, \quad m_3 = \frac{y_D^2 v^2}{3b-a}.$$
Very heavy flavon parameters $a, b \to$ very light neutrinos.
- mass sum rule:
$$\frac{1}{m_3} = \frac{1}{m_1} - \frac{2}{m_2}.$$
which is a prediction of the A_4 model! Be careful: a and b are complex.
- both NO and IO are possible; the sum rule implies a lower bound!

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Other symmetry groups

This is a typical symmetry-based recipe:

- pick up G, select irreps for L, ℓ_R, ν_R, add flavons at will;
- choose flavon vev alignment among possible choices;
- calculate M_ℓ, m_D, $M_R \to$ calculate $\mathcal{M}_\nu$;
- (analytically) diagonalize M_ℓ, $\mathcal{M}_\nu \to$ derive U_{PMNS};
- derive sum rule for $m_{1,2,3}$.

Many series of finite groups have been studied [Holthausen, Lim, Lindner, 2012] and some are close to the experimental PMNS matrix.