Symmetries and Mass Degeneracies in the Scalar Sector

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Outline

1. Flashback to 2012: Can mass-degenerate scalars of the 2HDM explain the $h \to \gamma\gamma$ anomaly (which has since disappeared)?

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   - Natural scalar mass degeneracies: a symmetry based approach

4. New features of mass degenerate scalars in the 3HDM

   - The replicated IDM
   - The Ivanov-Silva model (and the significance of CP4)

5. Final comments and open questions

based on: H.E. Haber, O.M. Ogreid, P. Osland and M.N. Rebelo, arXiv:1808.08629.

Flashback to 2012: Can Mass-degenerate scalars explain the $h \to \gamma\gamma$ anomaly?

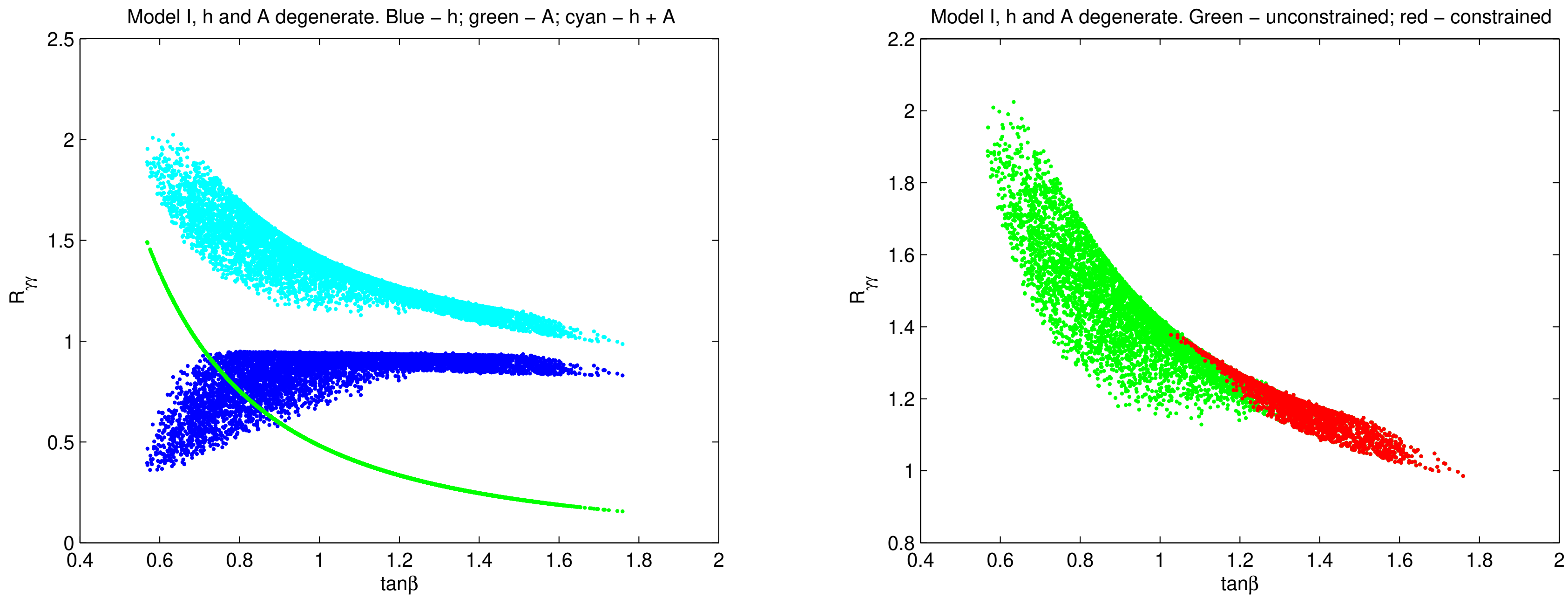
After the initial discovery of the Higgs boson in 2012, it appeared that the signal strength for $h \to \gamma\gamma$ was significantly enhanced above Standard Model (SM) expectations.

My collaborators and I proposed to explain this anomaly under the assumption that the the observed Higgs state at 125 GeV was in fact a pair of mass degenerate scalars.*

We considered the Type-I and Type-II two-Higgs doublet model (2HDM), and explored various possibilities for mass degeneracy and their phenomenological consequences.

*P.M. Ferreira, R. Santos, H.E. Haber and J.P. Silva, Phys. Rev. D **87**, 055009 (2013) [arXiv:1211.3131].

An enhanced $\gamma\gamma$ signal due to mass-degenerate h^0 and A^0:



Left panel: $R_{\gamma\gamma}$ as a function of $\tan\beta$ for h (blue), A (green), and the total observable rate (cyan), obtained by summing the rates with intermediate h and A, for the unconstrained scenario (i.e., the effects of virtual charged Higgs exchange in B physics is neglected). Right panel: Total rate for $R_{\gamma\gamma}$ as a function of $\tan\beta$ for the constrained (red) and unconstrained (green) scenarios.
Above, $R_f^H = \frac{\sigma(pp\to H)_{\rm 2HDM}\times {\rm BR}(H\to f)_{\rm 2HDM}}{\sigma(pp\to h_{\rm SM})\times {\rm BR}(h_{\rm SM}\to f)}$, where f is the final state of interest, and H is one of the two 125 GeV mass-degenerate scalars. The observed ratio of f production relative to the SM expectation is $R_f \equiv \sum_H R_f^H$. In our analysis, we assumed that $R_{WW} \simeq R_{ZZ} \simeq 1 \pm 0.2$.

The corresponding results in the Type-II 2HDM were similar. Other degenerate-mass scalar pairs were also considered. By the end of Run I of the LHC, the $\gamma\gamma$ excess was gone, and the Higgs data appears to be consistent with SM expectations.

Mass Degeneracies in Extended Higgs Sectors

We would like to explore the possibility of mass-degenerate neutral scalars and/or mass-degenerate charged Higgs pairs that can arise in extended Higgs sectors. In each case, the critical questions to ask are:

- Is the origin of the mass degeneracy natural? (Yes, if due to a symmetry. No, if accidental.)

- Can mass degenerate scalars be distinguished experimentally on an event by event basis?

- Is the only experimental signal of the scalar mass degeneracy a measurable multiplicity factor that arises when averaging over initial state degeneracies and summing over final state degeneracies?

A simple model of scalar mass degeneracy: $H^\pm$

Any doublet extended Higgs model has a mass degenerate state—the charged Higgs boson, $H^\pm$. Indeed, H^+ and H^- are degenerate due to the U(1)$_{\rm EM}$ gauge symmetry. Moreover, the H^+ and H^- are distinguishable by their electric charge, which we can probe using photons.

Suppose that this probe was unavailable (or equivalently, suppose one could turn off electromagnetism). Can experiment reveal the existence of a mass-degenerate scalar?

- Given a charged Higgs state, one could not physically distinguish between the two degenerate states.

- However, there would in principle be observables that are sensitive to the number of degenerate states present. Examples: $H \to H^+H^-$ (but not $Z \to H^+H^-$ due to the off-diagonal nature of this coupling).

Natural scalar mass degeneracies in the 2HDM

Consider the 2HDM with two hypercharge-one, doublet scalar fields. It is convenient to work in the Higgs basis in which the two Higgs doublet fields, denoted by H_1 and H_2, satisfy $\langle H_1^0\rangle = v/\sqrt{2}$ and $\langle H_2^0\rangle = 0$ (i.e., the vacuum expectation value, $v = 246$ GeV, resides entirely in the neutral component of the Higgs basis field H_1.)

We can immediately identify the physical charged Higgs field, $H^+ \equiv H_2^+$, and the neutral and charged Goldstone fields, $G^0 = \sqrt{2}\,\mathrm{Im}\,H_1^0$ and $G^+ \equiv H_1^+$. In the Higgs basis, the scalar potential is given by:

$$
\begin{aligned}
\mathcal{V} = {} & Y_1 H_1^\dagger H_1 + Y_2 H_2^\dagger H_2 + [Y_3 H_1^\dagger H_2 + \mathrm{h.c.}] + \tfrac{1}{2}Z_1(H_1^\dagger H_1)^2 \\
& + \tfrac{1}{2}Z_2(H_2^\dagger H_2)^2 + Z_3(H_1^\dagger H_1)(H_2^\dagger H_2) + Z_4(H_1^\dagger H_2)(H_2^\dagger H_1) \\
& + \left\{\tfrac{1}{2}Z_5(H_1^\dagger H_2)^2 + \big[Z_6(H_1^\dagger H_1) + Z_7(H_2^\dagger H_2)\big] H_1^\dagger H_2 + \mathrm{h.c.}\right\},
\end{aligned}
$$

where Y_1, Y_2 and $Z_{1,2,3,4}$ are real, whereas Y_3, $Z_{5,6,7}$ are potentially complex. After minimizing the scalar potential, $Y_1 = -\tfrac{1}{2}Z_1 v^2$ and $Y_3 = -\tfrac{1}{2}Z_6 v^2$.

Specializing to the Inert doublet model (IDM)

Suppose that the Higgs basis of the 2HDM exhibits an exact $\mathbb{Z}_2$ symmetry, $H_1 \to +H_1$ and $H_2 \to -H_2$. This symmetry is also preserved by the vacuum. It then follows that $Y_3 = Z_6 = Z_7 = 0$. The one remaining complex parameter, Z_5 can be chosen real by rephasing the Higgs basis field H_2. Thus, the IDM scalar potential is CP-conserving.

The Higgs basis doublet fields are also mass eigenstate fields,

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left[v + h + iG^0\right] \end{pmatrix}, \qquad H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left[H + iA\right] \end{pmatrix},$$

where $G^\pm$ and G^0 are the Goldstone bosons that provide the longitudinal degrees of freedom of the massive $W^\pm$ and Z^0 gauge bosons. The tree-level properties of the scalar h are precisely those of the SM Higgs boson. The physical scalar mass spectrum is,

$$m_h^2 = Z_1 v^2\,, \qquad m_{H^\pm}^2 = Y_2 + \tfrac{1}{2} Z_3 v^2\,,$$

$$m_A^2 = m_{H^\pm}^2 + \tfrac{1}{2}(Z_4 - Z_5)v^2\,, \qquad m_H^2 = m_A^2 + Z_5 v^2\,.$$

Scalar/vector Couplings of the IDM

$$\mathscr{L}_{VVH} = \left(g m_W W_\mu^+ W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu\right) h\,,$$

$$\begin{aligned}
\mathscr{L}_{VVHH} = & \left[\tfrac{1}{4}g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{8c_W^2} Z_\mu Z^\mu\right](h^2 + H^2 + A^2) \\
& + \left[\tfrac{1}{2}g^2 W_\mu^+ W^{\mu\,-} + e^2 A_\mu A^\mu + \frac{g^2}{c_W^2}\left(\tfrac{1}{2} - s_W^2\right)^2 Z_\mu Z^\mu + \frac{2ge}{c_W}\left(\tfrac{1}{2} - s_W^2\right) A_\mu Z^\mu\right] H^+ H^- \\
& + \left\{\left(\tfrac{1}{2} e g A^\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^+\right) H^-(H + iA) + \text{h.c.}\right\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{VHH} = & \frac{g}{2c_W} Z^\mu A \overleftrightarrow{\partial}_\mu H - \tfrac{1}{2}g\left[i W_\mu^+ H^- \overleftrightarrow{\partial}^\mu (H + iA) + \text{h.c.}\right] \\
& + \left[ieA^\mu + \frac{ig}{c_W}\left(\tfrac{1}{2} - s_W^2\right) Z^\mu\right] H^+ \overleftrightarrow{\partial}_\mu H^-,
\end{aligned}$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$.

The cubic and quartic Higgs self-interactions are governed by

$$\mathscr{L}_{3h} = -\tfrac{1}{2}v\left[Z_1 h^3 + (Z_3 + Z_4)h(H^2 + A^2) + Z_5 h(H^2 - A^2)\right] - v Z_3 h H^+ H^-\,.$$

$$\begin{aligned}
\mathscr{L}_{4h} = & -\tfrac{1}{8}\left[Z_1 h^4 + Z_2(H^2 + A^2)^2 + 2(Z_3 + Z_4)h^2(H^2 + A^2) + 2Z_5 h^2(H^2 - A^2)\right] \\
& -\tfrac{1}{2}H^+ H^-\left[Z_2(H^2 + A^2 + H^+ H^-) + Z_3 h^2\right].
\end{aligned}$$

A natural mass degeneracy of the IDM

$m_H = m_A$, due to $Z_5 = 0$.

This mass degeneracy is due to an exact continuous U(1) symmetry, $H_1 \to H_1$ and $H_2 \to e^{i\theta} H_2$, which is preserved by the vacuum. One can now define eigenstates of U(1) charge (not to be confused with electric charge),

$$\phi^\pm = \frac{1}{\sqrt{2}} \left[H \pm iA \right] .$$

The physical scalar mass spectrum of the mass-degenerate IDM is,

$$\begin{aligned} m_h^2 &= Z_1 v^2 , \\ m_{H^\pm}^2 &= Y_2 + \tfrac{1}{2} Z_3 v^2 , \\ m_{\phi^\pm}^2 &= Y_2 + \tfrac{1}{2}(Z_3 + Z_4) v^2 . \end{aligned}$$

Remark: If $Z_4 = 0$, then the $H^\pm$ are degenerate in mass with the $\phi^\pm$ at tree-level. But, this mass-degeneracy is broken by radiative corrections (due to the interactions with gauge bosons).

The relevant interaction terms of $\phi^\pm$ are

$$
\begin{aligned}
\mathscr{L}_{\rm int} = & \left[\tfrac{1}{2}g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{4c_W^2} Z_\mu Z^\mu\right]\phi^+\phi^- + \frac{ig}{2c_W} Z^\mu \phi^- \overleftrightarrow{\partial}_\mu\,\phi^+ - \frac{g}{\sqrt{2}}\left[iW_\mu^+ H^- \overleftrightarrow{\partial}^\mu\,\phi^+ + \text{h.c.}\right] \\
& + \frac{eg}{\sqrt{2}}\left(A^\mu W_\mu^+ H^- \phi^+ + A^\mu W_\mu^- H^+ \phi^-\right) - \frac{g^2 s_W^2}{\sqrt{2}c_W}\left(Z^\mu W_\mu^+ H^- \phi^+ + Z^\mu W_\mu^- H^+ \phi^-\right) \\
& - v(Z_3 + Z_4) h \phi^+ \phi^- - \tfrac{1}{2}\left[Z_2(\phi^+\phi^-)^2 + (Z_3+Z_4)h^2\phi^+\phi^-\right] - Z_2 H^+ H^- \phi^+ \phi^- \,.
\end{aligned}
$$

Although $\phi^\pm$ are mass degenerate states, they can be physically distinguished on an event by event basis.

For example, Drell-Yan production via a virtual s-channel W^+ exchange can produce H^+ in association with ϕ^-, whereas virtual s-channel W^- exchange can produce H^- in association with ϕ^+. Thus, the sign of the charged Higgs boson reveals the U(1)-charge of the produced neutral scalar. The origin of this correlation lies in the fact that, by construction, H^+ and ϕ^+ both reside in H_2, whereas H^- and ϕ^- both reside in $H_2^\dagger$.

Accidental mass degeneracies of the IDM

1. $m_h = m_H$ ($m_h = m_A$)

This degeneracy (at tree-level) occurs when $Z_1 v^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4 \pm Z_5)v^2$, respectively. This is an unnatural relation. In particular, the hh two-point function receives one-loop corrections from WW and ZZ loops, whereas the HH (AA) two-point functions receives one-loop corrections from AZ (HZ) loops.

The mass-degenerate states are physically distinguishable since h is $\mathbb{Z}_2$-even, whereas H (A) is $\mathbb{Z}_2$-odd. In particular, WW and ZZ fusion produces h but not H (A), whereas Drell-Yan production via a virtual Z couples only to HA.

2. $m_h = m_H = m_A$

Putting $Z_5 = 0$, we have $Z_1 v^2 = Y_2 + \frac{1}{2}(Z_3 + Z_4)v^2$. This is still an unnatural relation for the same reason given above.

Mass degeneracies in the most general 2HDM

It is also possible to construct examples of accidental mass degeneracies in the most general 2HDM. However, the $only$ natural neutral scalar mass degeneracy in the 2HDM is precisely the case of the IDM with $Z_5 = 0$.

To reach this conclusion, note the following remarkable result in the 2HDM. Denoting the three neutral scalar masses by m_1, m_2 and m_3, and their respective couplings to W^+W^- by e_1, e_2 and e_3, then

$$\mathrm{Im}(Z_5^* Z_6^2) = \frac{2e_1 e_2 e_3}{v^9}(m_1^2 - m_2^2)(m_1^2 - m_3^2)(m_2^2 - m_3^2)\,.$$

Thus, given any neutral scalar mass degeneracy, it is possible to find a Higgs basis in which Z_5 and Z_6 are simultaneously real. In this basis, the neutral scalar squared-mass matrix is block diagonal with a 2×2 block and a 1×1 block. The resulting expressions for the scalar masses then have simple analytic forms, and all possible mass-degenerate cases are easily analyzed.

The resulting Higgs mass relations are,

$$m^2_{H^\pm} = Y_2 + \tfrac{1}{2}Z_3 v^2\,, \qquad\qquad m^2_A = m^2_{H^\pm} + \tfrac{1}{2}(Z_4 - Z_5)v^2\,,$$
$$m^2_{H,h} = \frac{1}{2}\left\{m^2_A + (Z_1 + Z_5)v^2 \pm \sqrt{[m^2_A - (Z_1 - Z_5)v^2]^2 + 4Z_6^2 v^4}\right\}.$$

Mass degenerate neutral scalars arise if

$$Z_5(m^2_A - Z_1 v^2) + Z_6^2 v^2 = 0 \quad \text{or} \quad \left[m^2_A - (Z_1 - Z_5)v^2\right]^2 + 4Z_6^2 v^4 = 0\,.$$

Example 1: $m_h = m_H$

It follows that $m^2_A = (Z_1 - Z_5)v^2$ and $Z_6 = 0$. Thus, we recover the IDM mass spectrum for this degenerate case, although Z_7 can be nonzero, in which case we have two unnatural conditions.

Example 2: $m_h = m_A$ or $m_H = m_A$

Both these possibilities arise when $Z_5(m^2_A - Z_1 v^2) + Z_6^2 v^2 = 0$, which is an unnatural condition (unless $Z_5 = Z_6 = Z_7 = 0$).

Natural 2HDM scalar mass degeneracies revisited

Our previous analysis was based on examining mass degeneracies and then determining whether an underlying symmetry was responsible. An alternative approach is to start with a classification of possible symmetries and then check whether any of them yield mass degenerate scalar state.

The classification of 2HDM symmetries is known. We begin with possible symmetries of the 2HDM scalar Lagrangian (where the scalar field kinetic energy terms include gauge covariant derivatives).

symmetry	transformation law		
$\mathbb{Z}_2$	$\Phi_1 \to \Phi_1$	$\Phi_2 \to -\Phi_2$	
U(1)	$\Phi_1 \to \Phi_1$	$\Phi_2 \to e^{2i\theta}\Phi_2$	
SO(3)	$\Phi_a \to U_{ab}\Phi_b$	$U \in \mathrm{U}(2)/\mathrm{U}(1)_{\mathrm{Y}}$	(for $a,\ b = 1, 2$)
GCP1	$\Phi_1 \to \Phi_1^\dagger$	$\Phi_2 \to \Phi_2^\dagger$	
GCP2	$\Phi_1 \to \Phi_2^\dagger$	$\Phi_2 \to -\Phi_1^\dagger$	
GCP3	$\Phi_1 \to \Phi_1^\dagger \cos\theta + \Phi_2^\dagger \sin\theta$	$\Phi_2 \to -\Phi_1^\dagger \sin\theta + \Phi_2^\dagger \cos\theta$	(for $0 < \theta < \frac{1}{2}\pi$)
Π_2	$\Phi_1 \to \Phi_2$	$\Phi_2 \to \Phi_1$	

Imposing the various symmetries on the 2HDM scalar potential,

$$\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger H_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)$$
$$+ \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \left\{ \tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \left[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)\right] \Phi_1^\dagger \Phi_2 + \text{h.c.} \right\},$$

yields the following constraints on the scalar potential parameters:

symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	$\mathrm{Re}\,\lambda_5$	$\mathrm{Im}\,\lambda_5$	λ_6	λ_7
$\mathbb{Z}_2$	-	-	0	-	-	-	-	-	-	0	0
U(1)	-	-	0	-	-	-	-	0	0	0	0
SO(3)	-	m_{11}^2	0	-	λ_1	-	$\lambda_1 - \lambda_3$	0	0	0	0
GCP1	-	-	real	-	-	-	-	-	0	real	real
GCP2	-	m_{11}^2	0	-	λ_1	-	-	-	-	-	$-\lambda_6$
GCP3	-	m_{11}^2	0	-	λ_1	-	-	$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0
Π_2	-	m_{11}^2	real	-	λ_1	-	-	-	0	-	λ_6^*
$\mathbb{Z}_2 \oplus \Pi_2$	-	m_{11}^2	0	-	λ_1	-	-	-	0	0	0
U(1)$\oplus\Pi_2$	-	m_{11}^2	0	-	λ_1	-	-	0	0	0	0

Taken from: P.M. Ferreira, H.E. Haber and J.P. Silva, Phys. Rev. D **79**, 116004 (2009) [arXiv:0902.1537].

Remark: GCP2 is equivalent to $\mathbb{Z}_2 \oplus \Pi_2$ in another scalar field basis. Likewise, GCP3 is equivalent to U(1)$\oplus\Pi_2$ in another scalar field basis.

2HDM symmetries that do not produce scalar mass degeneracies

1. Imposing the symmetries, $\mathbb{Z}_2$, GCP1 or GCP2 does not generically yield scalar mass degeneracies.

2. Imposing a U(1) symmetry (this is the famous Peccei-Quinn symmetry) that is broken by the vacuum yields a massless Goldstone boson,and no scalar mass degeneracies for generic parameter choices.

3. Imposing a GCP3 symmetry and analyzing the scalar potential minimum conditions yields two classes of allowed vacua, $\langle\Phi_1^0\rangle = v_1$ and $\langle\Phi_2^0\rangle = v_2 e^{i\xi}$,
   A. $\sin\xi = 0$ and β arbitrary $(0 < \beta < \frac{1}{2}\pi)$,
   B. $\cos\xi = 0$ and $\cos 2\beta = 0$,

   where $\tan\beta \equiv v_2/v_1$. In each case, we can determine the scalar potential parameters in the Higgs basis. In the Class A vacuum, the GCP3 is realized in the Higgs basis (which is necessarily broken by the vacuum), which results in a massless Goldstone boson and no scalar mass degeneracies for generic parameter choices.

2HDM symmetries that yield scalar mass degeneracies

1. Imposing a U(1) symmetry in the Higgs basis (which is unbroken by the vacuum) yields the IDM with $Z_5 = 0$. The result is a mass degenerate pair H, A that reside in the inert doublet.

2. Imposing a GCP3 symmetry, we find that in the Class B vacuum the U(1)$\oplus\Pi_2$ symmetry is realized in the Higgs basis. This yields the IDM with $Z_5 = 0$ and two extra conditions, $Y_1 = Y_2$ and $Z_1 = Z_2$. This is a special case of Case 1.

3. Imposing an SO(3) symmetry (which is necessarily broken by the vacuum) yields two Goldstone bosons, H and A. This is a special case of Case 1.

If we now include the Higgs-fermion Yukawa couplings, then in Case 1 above, the U(1) symmetry remains unbroken if one employs Type-I Yukawa couplings (where the fermions do not couple to the inert doublet). However, in Cases 2 and 3, the corresponding symmetries are broken by the Yukawa couplings, in which case the resulting mass degeneracies will be spoiled.

New features of mass degenerate scalars in the 3HDM

In the 3HDM, one can now consider mass-degenerate charged Higgs pairs, as well as mass-degenerate neutral scalars. I will focus on three special 3HDMs where natural mass degeneracies occur.

The replicated IDM (RIDM)

We begin with a replicated IDM, in which two inert doublets are mass-degenerate. Consider the following 3HDM scalar potential in the Higgs basis,

$$
\begin{aligned}
\mathcal{V}_{\rm RIDM} = {} & Y_1 H_1^\dagger H_1 + Y_2\left(H_2^\dagger H_2 + H_3^\dagger H_3\right) + \tfrac{1}{2}Z_1(H_1^\dagger H_1)^2 + \tfrac{1}{2}Z_2(H_2^\dagger H_2 + H_3^\dagger H_3)^2 \\
& + Z_3(H_1^\dagger H_1)\left(H_2^\dagger H_2 + H_3^\dagger H_3\right) + Z_4\left[(H_1^\dagger H_2)(H_2^\dagger H_1) + (H_1^\dagger H_3)(H_3^\dagger H_1)\right] \\
& + \tfrac{1}{2}Z_5\left\{(H_1^\dagger H_2)^2 + (H_2^\dagger H_1)^2 + (H_1^\dagger H_3)^2 + (H_3^\dagger H_1)^2\right\}\,.
\end{aligned}
$$

Without loss of generality, we have chosen Z_5 real, so that $\mathcal{V}_{\rm RIDM}$ is CP-conserving. There is a continuous symmetry that is responsible for the mass-degeneracy of the inert Higgs doublets H_2 and H_3.

Consider the U(2) family symmetry, where the neutral complex field H_1^0 is a singlet and the neutral complex fields H_2^0 and H_3^0 transform as,

$$\begin{pmatrix} H_2^0 \\ H_3^0 \end{pmatrix} \longrightarrow U \begin{pmatrix} H_2^0 \\ H_3^0 \end{pmatrix}, \quad \text{with } U \in \mathrm{U}(2).$$

If $Z_5 = 0$, then $\mathcal{V}_{\rm RIDM}$ depends only on the combination of neutral fields, $H_2^{0\,\dagger} H_2^0 + H_3^{0\,\dagger} H_3^0$, and hence is invariant under U(2).

If $Z_5 \neq 0$, then $\mathcal{V}_{\rm RIDM}$ also depends on the combination of neutral fields, $(H_2^0)^2 + (H_2^{0\,\dagger})^2 + (H_3^0)^2 + (H_3^{0\,\dagger})^2$. Hence, $\mathcal{V}_{\rm RIDM}$ is invariant under an O(2) subgroup of the U(2) transformations (corresponding to real unitary matrices).

The O(2) symmetry guarantees that the real and imaginary parts of H_2^0 and H_3^0 are separately mass degenerate. In the case of $Z_5 = 0$ (and the full U(2) family symmetry), one has in addition a mass-degeneracy between the real and imaginary parts of each inert neutral scalar.

There is another continuous symmetry at play here, which takes the form of a generalized CP transformation (GCP),

$$\begin{pmatrix} H_2^0 \\ H_3^0 \end{pmatrix} \longrightarrow U \begin{pmatrix} H_2^{0\,\dagger} \\ H_3^{0\,\dagger} \end{pmatrix}, \quad \text{with } U \in \mathrm{U}(2)_{\rm GCP}.$$

Again, if $Z_5 = 0$, then $\mathcal{V}_{\rm RIDM}$ is invariant under the $\mathrm{U}(2)_{\rm GCP}$. If $Z_5 \neq 0$, then $\mathcal{V}_{\rm RIDM}$ is invariant under an $\mathrm{O}(2)_{\rm GCP}$ subgroup of $\mathrm{U}(2)_{\rm GCP}$.

Including the kinetic energy terms (with gauge covariant derivatives), the relevant global symmetry group associated with the mass-degenerate scalars is a semi-direct product, $\mathrm{O}(2)\rtimes\mathbb{Z}_2$ (which is enlarged to $\mathrm{U}(2)\rtimes\mathbb{Z}_2$ if $Z_5 = 0$).

Remark: The mass degeneracies of the inert *charged* Higgs scalars are governed by the full $\mathrm{U}(2)\rtimes\mathbb{Z}_2$ symmetry (since Z_5 does not contribute to the inert charged Higgs scalar masses).

In the replicated IDM, the Higgs basis doublet fields are mass eigenstate fields,

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left[v + h_{\rm SM} + iG^0\right] \end{pmatrix}, \quad H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left[H + iA\right] \end{pmatrix}, \quad H_3 = \begin{pmatrix} h^+ \\ \frac{1}{\sqrt{2}}\left[h + ia\right] \end{pmatrix},$$

with a minor change of notation from the IDM. The corresponding masses are,

$$m^2_{H^\pm} = m^2_{h^\pm} = Y_2 + \tfrac{1}{2}Z_3 v^2\,, \qquad m^2_H = m^2_h = Y_2 + \tfrac{1}{2}(Z_3 + Z_4 + Z_5)v^2\,,$$
$$m^2_A = m^2_a = Y_2 + \tfrac{1}{2}(Z_3 + Z_4 - Z_5)v^2\,.$$

The corresponding couplings simply replicate the IDM couplings. For example,

$$\mathscr{L}_{VVH} = \left(g m_W W^+_\mu W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu\right) h_{\rm SM}\,,$$

$$\mathscr{L}_{VHH} = \frac{g}{2c_W} Z^\mu (A \overleftrightarrow{\partial}_\mu H + a \overleftrightarrow{\partial}_\mu h) - \tfrac{1}{2}g\left[iW^+_\mu H^- \overleftrightarrow{\partial}^\mu (H + iA) + iW^+_\mu h^- \overleftrightarrow{\partial}^\mu (h + ia) + \text{h.c.}\right]$$
$$+ \left[ieA^\mu + \frac{ig}{c_W}\left(\tfrac{1}{2} - s^2_W\right) Z^\mu\right](H^+ \overleftrightarrow{\partial}_\mu H^- + h^+ \overleftrightarrow{\partial}_\mu h^-),$$

$$\mathscr{L}_{3h} = -\tfrac{1}{2}v\left[Z_1 h^3_{\rm SM} + (Z_3 + Z_4)h_{\rm SM}(H^2 + A^2 + h^2 + a^2) + Z_5 h_{\rm SM}(H^2 - A^2 + h^2 - a^2)\right]$$
$$- v Z_3 h_{\rm SM}(H^+ H^- + h^+ h^-)\,.$$

It is convenient to introduce,

$$P \equiv \frac{H + ih}{\sqrt{2}}\,, \qquad P^\dagger \equiv \frac{H - ih}{\sqrt{2}}\,, \qquad Q \equiv \frac{A - ia}{\sqrt{2}}\,, \qquad Q^\dagger \equiv \frac{A + ia}{\sqrt{2}}\,.$$

Then, we can rewrite the RIDM couplings in terms of the complex fields P, Q (and their adjoints). For example,

$$\begin{aligned}
\mathscr{L}_{VHH} = \frac{g}{2c_W}\,Z^\mu(Q\overleftrightarrow{\partial}_\mu P + Q^\dagger\overleftrightarrow{\partial}_\mu P^\dagger) - \frac{g}{2\sqrt{2}}\Big[&(iW_\mu^+ H^- - W_\mu^- h^+)\overleftrightarrow{\partial}^\mu (P + iQ) \\
&-(iW_\mu^- H^+ - W_\mu^+ h^-)\overleftrightarrow{\partial}^\mu (P - iQ) + \text{h.c.}\Big] \\
+ \left[ieA^\mu + \frac{ig}{c_W}\left(\tfrac{1}{2} - s_W^2\right) Z^\mu\right]&(H^+\overleftrightarrow{\partial}_\mu H^- + h^+\overleftrightarrow{\partial}_\mu h^-),
\end{aligned}$$

$$\mathscr{L}_{3h} = -v\left[\tfrac{1}{2}Z_1 h_{\rm SM}^3 + (Z_3 + Z_4)h_{\rm SM}(|P|^2 + |Q|^2) + Z_5 h_{\rm SM}(|P|^2 - |Q|^2)\right] - vZ_3 h_{\rm SM}(H^+H^- + h^+h^-).$$

In the RIDM, there is no experimental measurement that can physically distinguish the degenerate scalars, $(H^\pm, h^\pm)$, (H, h) and (A, a). However, the multiplicity factor will appear after summing over final mass-degenerate states, e.g., $Z \to HA$, ha (or equivalently, $Z \to PQ$, $P^\dagger Q^\dagger$), doubles the rate into a pair of neutral scalars.

Mass degeneracies beyond the RIDM

Naively, one can add to the RIDM scalar potential any gauge invariant quartic term involving the doublet fields H_2 and H_3 without spoiling the mass degeneracies of the RIDM. However, the resulting tree-level mass degeneracies will be unnatural unless they are a consequence of a symmetry.

The simplest possible modification of the RIDM is to remove the $(H_2^\dagger H_2)(H_3^\dagger H_3)$ term entirely from the scalar potential. That is, we can define a RIDM′ scalar potential as,

$$\begin{aligned}\mathcal{V}_{\mathrm{RIDM}'} &= \mathcal{V}_{\mathrm{RIDM}} - Z_2(H_2^\dagger H_2)(H_3^\dagger H_3) \\ &= \ldots + \tfrac{1}{2}Z_2\big[(H_2^\dagger H_2)^2 + (H_3^\dagger H_3)^2\big] + \ldots\end{aligned}$$

The mass degeneracies of the RIDM' are no longer a consequence of a continuous symmetry, which is now explicitly broken by the presence of the term in red above. Indeed, this term is invariant only under a discrete subgroup of O(2), which can be identified as the dihedral group D_4.

In more detail, the dihedral group is

$$D_4 \cong \{\mathbb{1}, -\mathbb{1}, R, -R, S, -S, Z, -Z\}\,,$$

where $\mathbb{1}$ is the 2×2 identity matrix and

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

We recognize D_4 as the dihedral group of order eight, which is the symmetry group of the square.

Extending our considerations to GCP symmetries, $\mathcal{V}_{\rm RIDM'}$ is also invariant under $(D_4)_{\rm GCP}$. Including the kinetic energy terms (with gauge covariant derivatives), the relevant global symmetry group associated with the mass-degenerate scalars of the RIDM' is a semi-direct product, $D_4 \rtimes \mathbb{Z}_2$.

Question: Can we break this discrete symmetry further (by adding more terms to $\mathcal{V}_{\rm RIDM'}$) while maintaining natural mass-degenerate scalars?

The Ivanov-Silva Model

Ivanov and Silva (IS) introduced a particular 3HDM model with some curious properties.† In the Higgs basis of the 3HDM, we are free to make an arbitrary U(2) rotation to define the Higgs basis fields, H_2 and H_3. We have made use of this freedom to make a minor alteration of the IS scalar potential,

$$\begin{aligned}\mathcal{V}_{\rm IS} = \mathcal{V}_{\rm RIDM} &+ Z_3'(H_2^\dagger H_2)(H_3^\dagger H_3) + Z_4'(H_2^\dagger H_3)(H_3^\dagger H_2) \\ &+ \left[Z_8(H_2^\dagger H_3)^2 + Z_9(H_2^\dagger H_3)(H_2^\dagger H_2 - H_3^\dagger H_3) + {\rm h.c.}\right],\end{aligned}$$

where $\mathcal{V}_{\rm RIDM}$ is the replicated IDM scalar potential, and Z_8 and Z_9 are potentially complex.

The IS model still yields mass-degenerate inert doublets, since none of the extra terms involve the Higgs basis field H_1. Hence, these terms do not contribute to the tree-level scalar squared-mass matrices.

†I.P. Ivanov and J.P. Silva, Phys. Rev. D **93**, 095014 (2016) [arXiv:1512.09276],

Symmetries governing the mass degeneracies of the IS model

Note that after the extra terms in the scalar potential are included, there is no remaining unbroken continuous subgroup of the U(2) family symmetry or the $\mathrm{U(2)_{GCP}}$ generalized CP symmetry.

Case 1: Z_8 and Z_9 are real.

$\mathcal{V}_{\rm IS}$ is invariant under a discrete $\mathbb{Z}_4$ subgroup of the U(2) family symmetry group. The elements of this subgroup are,

$$\mathbb{Z}_4 = \{\mathbb{1}, -\mathbb{1}, Z, -Z\}\,, \quad \text{where } Z \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

where the 2×2 matrices above act on the Higgs basis fields H_2 and H_3. Note that $Z^2 = -\mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix.

The fields H_2 and H_3 are odd under $-\mathbb{1}$, which simply identifies the two inert doublets. The elements Z (and $-Z$) act non-trivially on the inert doublets.

As before, we are free to combine mass-degenerate neutral fields and define,

$$P \equiv (H + ih)/\sqrt{2} \quad \text{and} \quad Q \equiv (A - ia)/\sqrt{2}\,,$$

which are eigenstates of Z (and $-Z$). Indeed, P and $Q^\dagger$ have eigenvalue i under Z, and $P^\dagger$ and Q have eigenvalue $-i$ under Z. For example, this is consistent with the couplings of neutral scalars to the Z, namely

$$\mathscr{L}_{ZHH} = \frac{g}{2c_W} Z^\mu (P \overleftrightarrow{\partial}_\mu Q + P^\dagger \overleftrightarrow{\partial}_\mu Q^\dagger)\,.$$

Likewise, $\mathcal{V}_{\rm IS}$ is invariant under a discrete $\mathbb{Z}_4$ subgroup of the U(2)$_{\rm GCP}$ generalized CP symmetry, The element Z involved in the transformation,

$$\begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_2^\dagger \\ H_3^\dagger \end{pmatrix}\,,$$

is called a CP4 transformation by Ivanov and Silva.‡ Due to the extra dagger, P and Q have eigenvalue i and $P^\dagger$ and $Q^\dagger$ have eigenvalue $-i$ under Z. This is again consistent with the form of $\mathscr{L}_{ZHH}$ above since the Z is CP-even and parity introduces an extra minus sign due to the space derivative.

‡Note that $({\rm CP4})^2 \neq \mathbb{1}$ and $({\rm CP4})^4 = \mathbb{1}$. Hence the nomenclature.

Either discrete symmetry (family or GCP) can be invoked to explain the observed mass degeneracies of the IS model with real Z_8 and Z_9. Moreover, the conventional CP, called CP2 [since $(\mathrm{CP2})^2 = \mathbb{1}$], corresponding to $H_i \to H_i^\dagger$, is a symmetry since all scalar potential parameters are real.

<u>Case 2</u>: Z_8 and/or Z_9 are complex.

In this case, the symmetry transformation,

$$\begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_2 \\ H_3 \end{pmatrix},$$

is no longer respected by $\mathcal{V}_{\rm IS}$. The remaining unbroken family symmetry is $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$, which protects the inertness of H_2 and H_3 but cannot enforce the mass degeneracies of the IS model.

Nevertheless, the CP4 symmetry remains intact and is ultimately responsible for the IS model mass degeneracies. Under the assumption that Z_5, Z_8 and Z_9 are all nonzero, then one can show that there is no possible change of basis in which all scalar potential parameters are real, i.e. CP2 is not a symmetry.

A physical distinction between the CP2 and CP4 symmetry

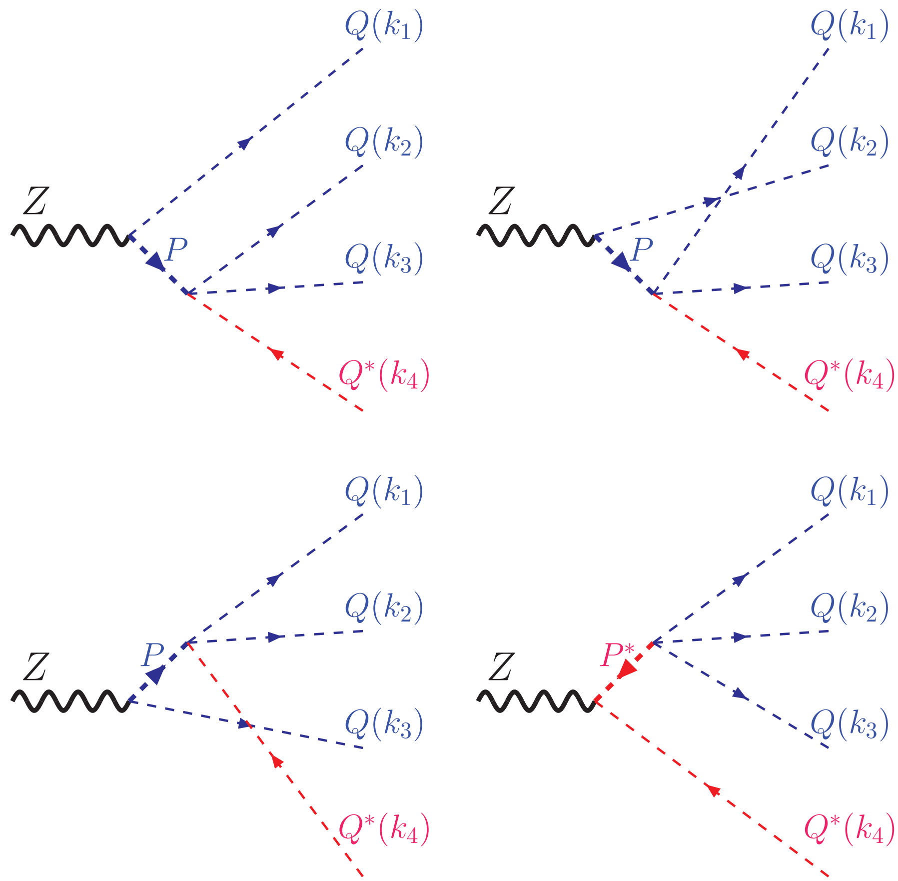
Ivanov and Silva asked: is there an experiment that can determine the order of the CP symmetry of the IS scalar sector? The answer is affirmative. It relies on the existence of a particular four scalar coupling of the IS model,

$$\begin{aligned}\delta\mathscr{L}_{4h} \ni \tfrac{1}{2}\,\mathrm{Im}\,Z_8\big[(PQ - P^\dagger Q^\dagger)(P^2 - Q^2 - P^{\dagger\,2} + Q^{\dagger\,2})\big] \\ + \tfrac{1}{2} i\,\mathrm{Im}\,Z_9\big[(PQ - P^\dagger Q^\dagger)(P^2 + Q^2 + P^{\dagger\,2} + Q^{\dagger\,2})\big]\,.\end{aligned}$$

Self-interaction terms of this type are absent if Z_8 and Z_9 are both real. As an example, consider the case where $M_Q \ll m_Z$ and $M_P \gg m_Z$. In this case, the four-scalar interactions above mediate the four body Z decay,

$$Z \to QQQQ^*\,, \quad QQ^*Q^*Q^*\,.$$

These two final states are experimentally indistinguishable, so we must sum incoherently the squared amplitudes of both channels. Observation of such decays would be consistent with the presence of a CP4 symmetry and would force us to conclude that it is impossible to define CP as a CP2 symmetry.



We have obtained (for $M_P \gg m_Z$ and $M_Q = 0$),

$$\frac{\Gamma(Z \to QQQQ^*, QQ^*Q^*Q^*)}{\Gamma(Z \to \nu\bar{\nu})} = \frac{2\left[(\mathrm{Im}\, Z_8)^2 + (\mathrm{Im}\, Z_9)^2\right]}{3 \cdot 5 \cdot 2^9\, \pi^4} \left(\frac{m_Z}{M_P}\right)^4 ,$$

whee the factor of 2 accounts for the multiplicity of mass-degenerate states.

An invariant distinction between CP4 with and without CP2

The form of the IS scalar potential used so far is basis dependent, even within the class of Higgs bases. But, there is a subset of Higgs bases, in which the IS scalar potential is applicable. Within this subset of Higgs bases, $(\mathrm{Im}\, Z_8)^2 + (\mathrm{Im}\, Z_9)^2$ must be a physical quantity, which means that one cannot find another basis within this subset such that both Z_8 and Z_9 are real.

Can we do better? Indeed, there exists a scalar basis invariant quantity that reduces to $(\mathrm{Im}\, Z_8)^2 + (\mathrm{Im}\, Z_9)^2$ in the subset of Higgs bases where the scalar potential is of the IS form.

Consider the 3HDM scalar potential in an arbitrary scalar field basis with a $\mathrm{U}(1)_{\mathrm{EM}}$ preserving minimum,

$$\mathcal{V} = Y_{a\bar{b}}\Phi_{\bar{a}}^{\dagger}\Phi_b + \tfrac{1}{2}Z_{a\bar{b}c\bar{d}}(\Phi_{\bar{a}}^{\dagger}\Phi_b)(\Phi_{\bar{c}}^{\dagger}\Phi_d)\,,$$

where $Z_{a\bar{b}c\bar{d}} = Z_{c\bar{d}a\bar{b}}$, subject to the hermiticity conditions, $Y_{a\bar{b}} = (Y_{b\bar{a}})^*$ and $Z_{a\bar{b}c\bar{d}} = (Z_{b\bar{a}d\bar{c}})^*$. The neutral Higgs vacuum expectation values are, $\langle\Phi_a^0\rangle = v\widehat{v}_a/\sqrt{2}$, where $v = 246$ GeV and $\widehat{v}_a$ is a vector of unit norm. It is convenient to define the hermitian matrix

$$V_{a\bar{b}} \equiv \widehat{v}_a\,\widehat{v}_{\bar{b}}^{*}\,.$$

Invariant quantities are constructed out of Y, Z and V such that all barred-unbarred index pairs are summed over.

A list of invariants and their values in the IS basis

$$J_1 \equiv V_{a\bar{c}} V_{b\bar{d}} Z_{c\bar{a}d\bar{b}}, \qquad J_2 \equiv V_{a\bar{b}} Z_{b\bar{a}c\bar{c}}, \qquad J_3 \equiv V_{a\bar{b}} Z_{b\bar{c}c\bar{a}},$$

$$J_4 \equiv V_{a\bar{b}} Z_{b\bar{d}c\bar{e}} Z_{d\bar{a}e\bar{c}}, \qquad J_5 \equiv V_{a\bar{b}} Z_{b\bar{d}c\bar{e}} Z_{d\bar{f}e\bar{g}} Z_{f\bar{a}g\bar{c}},$$

$$J_6 \equiv V_{a\bar{b}} Z_{b\bar{d}c\bar{e}} Z_{d\bar{f}e\bar{g}} Z_{f\bar{h}g\bar{k}} Z_{h\bar{a}k\bar{c}}.$$

In the IS basis, these invariants are given by,

$$J_1 = Z_1, \qquad J_2 = Z_1 + 2Z_3, \qquad J_3 = Z_1 + 2Z_4,$$

$$J_4 = Z_1^2 + 2Z_3^2 + 2Z_4^2 + 2Z_5^2,$$

$$J_5 = Z_1^3 + 4Z_5^2 Z_1 + 2Z_3^3 + 6Z_3 Z_4^2 + 2Z_2 Z_5^2 + 4Z_5^2 \,\mathrm{Re}\, Z_8,$$

$$J_6 = Z_1^4 + 2Z_3^4 + 2Z_4^4 + 12 Z_3^2 Z_4^2 + 4Z_5^4 + 2Z_5^2(3Z_1^2 + 2Z_1 Z_2 + Z_2^2)$$
$$+8Z_5^2\big[|Z_8|^2 + (Z_1 + Z_2)\,\mathrm{Re}\, Z_8 + (\mathrm{Im}\, Z_9)^2\big].$$

Note that Z_5 can be expressed in terms of an invariant quantity,

$$Z_5^2 = -J_1^2 + \tfrac{1}{2} J_1\,(J_2 + J_3) - \tfrac{1}{4}(J_2^2 + J_3^2) + \tfrac{1}{2} J_4\,.$$

Finally, we have discovered a remarkable invariant quantity,

$$
\begin{aligned}
\mathcal{N} = 32Z_5^2 J_6 - 16J_5^2 + 8J_5(3J_{21}J_{31}^2 + K) - J_{31}^4(9J_{21}^2 + 4Z_5^2) - 6KJ_{21}J_{31}^2 \\
-24Z_5^2 J_{21}^2 J_{31}^2 - J_{21}^6 - 4Z_5^2 J_{21}^4 - 8J_1(J_1^2 + 2Z_5^2)J_{21}^3 - 16J_1^6 \\
-96Z_5^2 J_1^4 - 192Z_5^4 J_1^2 - 128Z_5^6\,,
\end{aligned}
$$

where $J_{ij} \equiv J_i - J_j$ and $K \equiv 4J_1^3 + 8Z_5^2 J_1 + J_{21}^3$.

Plugging in the expressions for $J_1\,, \ldots\,, J_6$ given above, we find

$$
\mathcal{N} = 256Z_5^4\big[(\mathrm{Im}\,Z_8)^2 + (\mathrm{Im}\,Z_9)^2\big]\,.
$$

It follows that if $Z_5 \neq 0$ then there exists a ratio of invariant quantities, which when evaluated in the IS-basis, is equal to $(\mathrm{Im}\,Z_8)^2 + (\mathrm{Im}\,Z_9)^2$.

If $\mathcal{N} \neq 0$, then the CP4-conserving IS-potential does not respect a CP2 symmetry. If $\mathcal{N} = 0$, then the CP2 symmetry is respected, and a real Higgs basis exists (in which all the scalar potential parameters are real).

Special case of $Z_5 = 0$

If $Z_5 = 0$, then a real Higgs basis exists. How is this consistent with the previous computation of the decay rate for $Z \to QQQQ^*$, $QQ^*Q^*Q^*$? The resolution of this apparent paradox is that when $Z_5 = 0$, the masses of P and Q (and their complex conjugates) become degenerate. Hence, the Z decays into the Qs and Q^*s cannot be distinguished from similar decays where we substitute P for Q, etc.

The observable in this case corresponds to the incoherent sum of squared amplitudes for Z decay into four neutral scalars, summing over all possible combinations of P, Q, P^* and Q^* in the final state consistent with the corresponding CP4 quantum numbers. These amplitudes involve four scalar couplings that depend on other combinations of the scalar potential parameters.

The observable will thus be proportional to a more complicated combination of scalar potential parameters than $(\mathrm{Im}\, Z_8)^2 + (\mathrm{Im}\, Z_9)^2$, and must also be an invariant quantity (which is different from $\mathcal{N}$).

Proof of the existence of a real basis when $Z_5 = 0$

The most general basis transformation that preserves the general class of Higgs bases is given (in block diagonal form) by,

$$\begin{pmatrix} \bar{H}_1 \\ \bar{H}_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{V} \end{pmatrix} \begin{pmatrix} H_1 \\ H_{23} \end{pmatrix} ,$$

where

$$H_{23} \equiv \begin{pmatrix} H_2 \\ H_3 \end{pmatrix} , \qquad \bar{H}_{23} \equiv \begin{pmatrix} \bar{H}_2 \\ \bar{H}_3 \end{pmatrix} ,$$

and $\widetilde{V}$ is the most general U(2) matrix,

$$\widetilde{V} = e^{i\psi/2} \begin{pmatrix} e^{i\alpha}\cos\phi & -e^{-i\beta}\sin\phi \\ e^{i\beta}\sin\phi & e^{-i\alpha}\cos\phi \end{pmatrix} ,$$

where $0 \leq \phi < \pi$, $-\pi < \psi \leq \pi$, $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq \pi$. It is convenient to define,

$$\xi \equiv \alpha + \beta \,, \qquad \chi \equiv \alpha - \beta \,.$$

In the new scalar basis, the form of the CP4 symmetry transformation is modified. Written in terms of the barred scalar fields,

$$\bar{H}_i \to \bar{X}_{ij}\bar{H}_j^\dagger\,, \qquad \text{where } \bar{X} = VWV^T\,,$$

where the 3×3 matrices $\bar{X}$, V and W in block form are given by

$$\bar{X} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{X} \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{V} \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix},$$

and $\epsilon \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For $\widetilde{V} \in$ U(2) previously given, we have

$$\widetilde{X} = \widetilde{V}\epsilon\widetilde{V}^T = e^{i\psi}\epsilon\,,$$

after taking the determinant and noting that $\det\widetilde{V} = e^{i\psi}$.

In terms of the barred fields, the form of the IS potential is almost the same as before. The Z_5 term is modified as follows,

$$
\begin{aligned}
\mathcal{V}_{\rm IS} \ni\; & i\bar{Z}_5'\big[e^{i\psi}(\bar{H}_3^\dagger \bar{H}_1)(\bar{H}_2^\dagger \bar{H}_1) - e^{-i\psi}(\bar{H}_1^\dagger \bar{H}_2)(\bar{H}_1^\dagger \bar{H}_3)\big]\,, \\
& +\left\{\tfrac{1}{2}\bar{Z}_5\left[e^{i\psi}(\bar{H}_2^\dagger \bar{H}_1)^2 + e^{-i\psi}(\bar{H}_1^\dagger \bar{H}_3)^2\right] + {\rm h.c.}\right\},
\end{aligned}
$$

where[§]

$$
\begin{aligned}
\bar{Z}_5' &= Z_5 \sin 2\phi \sin\xi\,, \\
\bar{Z}_5 &= e^{i\chi} Z_5\big(e^{i\xi}\cos^2\phi + e^{-i\xi}\sin^2\phi\big)\,.
\end{aligned}
$$

Thus, if $Z_5 = 0$ then the only potential complex coefficients in the new basis (expressed in terms of the barred fields) are $\bar{Z}_8$ and $\bar{Z}_9$.

Note that if $\xi = \chi = 0$, then the original form of the IS potential is retained. We can use the remaining freedom to choose ϕ to obtain an IS basis in which Z_9 is real (and only Z_8 is potentially complex).

[§]Note that $|\bar{Z}_5|^2 + \bar{Z}_5'^{\,2} = Z_5^2$. The invariant quantity previously identified as Z_5^2 in the IS basis is given by $|\bar{Z}_5|^2 + \bar{Z}_5'^{\,2}$ in a generic Higgs basis.

Thus without loss of generality, we can assume that Z_9 is real and $Z_8 = |Z_8|e^{i\theta_8}$ is complex in the IS basis. We then perform the U(2) basis transformation given previously. In the new basis,

$$\mathrm{Im}\,\bar{Z}_8 = f_a \cos 2\chi - f_b \sin 2\chi\,, \qquad\qquad \mathrm{Im}\,\bar{Z}_9 = f_c \cos\chi - f_d \sin\chi\,,$$

where

$$f_a = |Z_8| \cos 2\phi \sin(2\xi + \theta_8) + Z_9 \sin 2\phi \sin\xi\,,$$

$$f_b = \tfrac{1}{4}(Z_3' + Z_4') \sin^2 2\phi - |Z_8|(1 - \tfrac{1}{2}\sin^2 2\phi) \cos(2\xi + \theta_8) - Z_9 \sin 2\phi \cos 2\phi \cos\xi\,,$$

$$f_c = -|Z_8| \sin 2\phi \sin(2\xi + \theta_8) + Z_9 \cos 2\phi \sin\xi\,,$$

$$f_d = \tfrac{1}{2}(Z_3' + Z_4') \sin 2\phi \cos 2\phi + |Z_8| \sin 2\phi \cos 2\phi \cos(2\xi + \theta_8) - Z_9 \cos 4\phi \cos\xi\,.$$

We now search for parameters of the U(2) basis transformation such that $\mathrm{Im}\,\bar{Z}_8 = \mathrm{Im}\,\bar{Z}_9 = 0$. Assuming that $f_a \neq 0$ and $f_c \neq 0$, it would then follow that

$$\cot\chi = \frac{f_d}{f_c}\,, \qquad\qquad \cot 2\chi = \frac{f_b}{f_a}\,.$$

Employing the trigonometric identity, $\cot 2\chi = (\cot^2\chi - 1)/(2\cot\chi)$, we conclude that $\mathrm{Im}\,\bar{Z}_8 = \mathrm{Im}\,\bar{Z}_9 = 0$ if and only if,

$$G(\phi,\xi) \equiv f_a(f_d^2 - f_c^2) - 2f_b f_c f_d = 0\,.$$

It is quite easy to check that $G(0,\xi) = -G(\tfrac{1}{2}\pi,\xi) = Z_9^2\,\mathrm{Im}\,Z_8$. Hence, for any choice of ξ, there must exist a value of ϕ between 0 and $\frac{1}{2}\pi$ such that $G(\phi,\xi) = 0$.

Thus, we have proven that if $Z_5 = 0$, then it is possible to find a new Higgs basis in which Z_8 and Z_9 are real. That is, a real Higgs basis exists (in which case CP2 is also a good symmetry of the model).

Remark: If $Z_5 \neq 0$, then it is still possible to find a new Higgs basis in which $\bar{Z}_8$ and $\bar{Z}_9$ are real. But, in this case, the complex parameters will reside in either $i\bar{Z}_5' e^{\pm i\psi}$ and/or $\bar{Z}_5 e^{\pm i\psi}$. That is, starting from the IS-basis where either Z_8 and/or Z_9 is complex, no real Higgs basis exists and CP2 is therefore not a symmetry of the model.

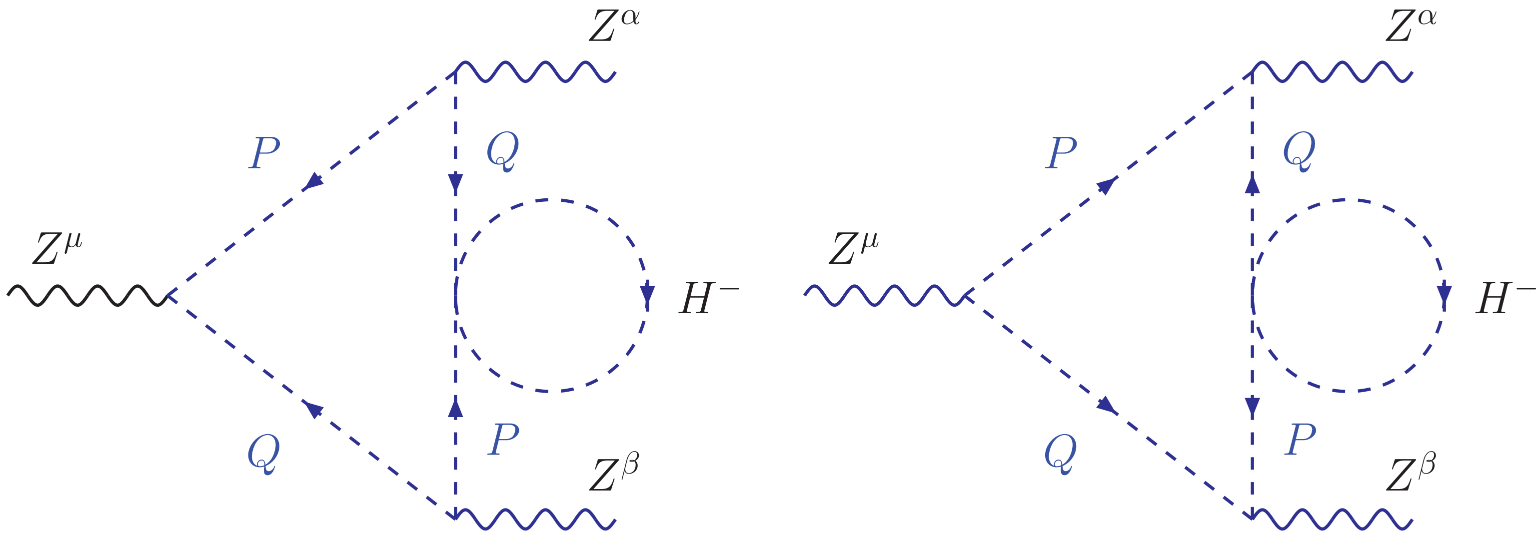
Final comments

1. For an N-Higgs doublet model, a CP4-symmetric scalar potential *and* CP4-invariant vacuum implies the existence of mass degenerate scalar states (similar to that of the IS model).

2. There is an observable distinction between CP4-invariant models that either respect or violate the conventional CP (CP2) symmetry.

3. CP4 invariant scalar sectors that violate CP2 can never yield any observable CP-violating phenomena, by defining CP acting on scalar fields to be CP4 and acting on gauge boson and fermion field to be the standard CP operator.

It is instructive to see how this works in practice. We examined the CP-violating form factors arising in the ZZZ and ZW^+W^- vertex, that would be radiatively generated due to the CP2-violating, CP4-conserving PQ^3 and P^3Q interactions.



One can see explicitly at two and three loops how such contributions to the CP-violating form factors vanish exactly (due to the absence of diagrams or diagrams canceling in pairs).

Denoting $\ell \equiv p_2 - p_3 \equiv 2p_2 - p_1$, the ZZZ vertex structure reduces to the form

$$-i\Gamma^{\alpha\beta\mu}_{ZZZ} = \frac{p_1^2 - M_Z^2}{M_Z^2}\left[f_4^Z(p_1^\alpha g^{\mu\beta} + p_1^\beta g^{\mu\alpha}) + f_5^Z \epsilon^{\mu\alpha\beta\rho}\ell_\rho\right] .$$

The dimensionless form factor f_4^Z violates CP while f_5^Z conserves CP.

Some open questions

1. In the IS model with a CP4-symmetric (but CP2-violating) scalar sector, one can couple fermions exclusively to H_1 (in Type-I fashion). But, then CP4 is not an exact symmetry since CP is violated due to the nonzero phase of the CKM matrix. Does this mean that the scalar mass degeneracies of the IS model are unnatural?

2. Neglecting the fermion couplings, can one prove that in the absence of scalar mass degeneracies, the existence of a CP symmetry in the scalar sector of an N Higgs doublet model always implies the existence of a real Higgs basis? (The answer is yes for $N = 2$. What about all $N > 2$?)

3. Are there any natural mass degeneracies of the 3HDM with a scalar potential that are distinct from that of the IS model (and special cases thereof)?

4 Is there a complete classification of all possible 3HDM scalar sector symmetries? (Partial results are known.)