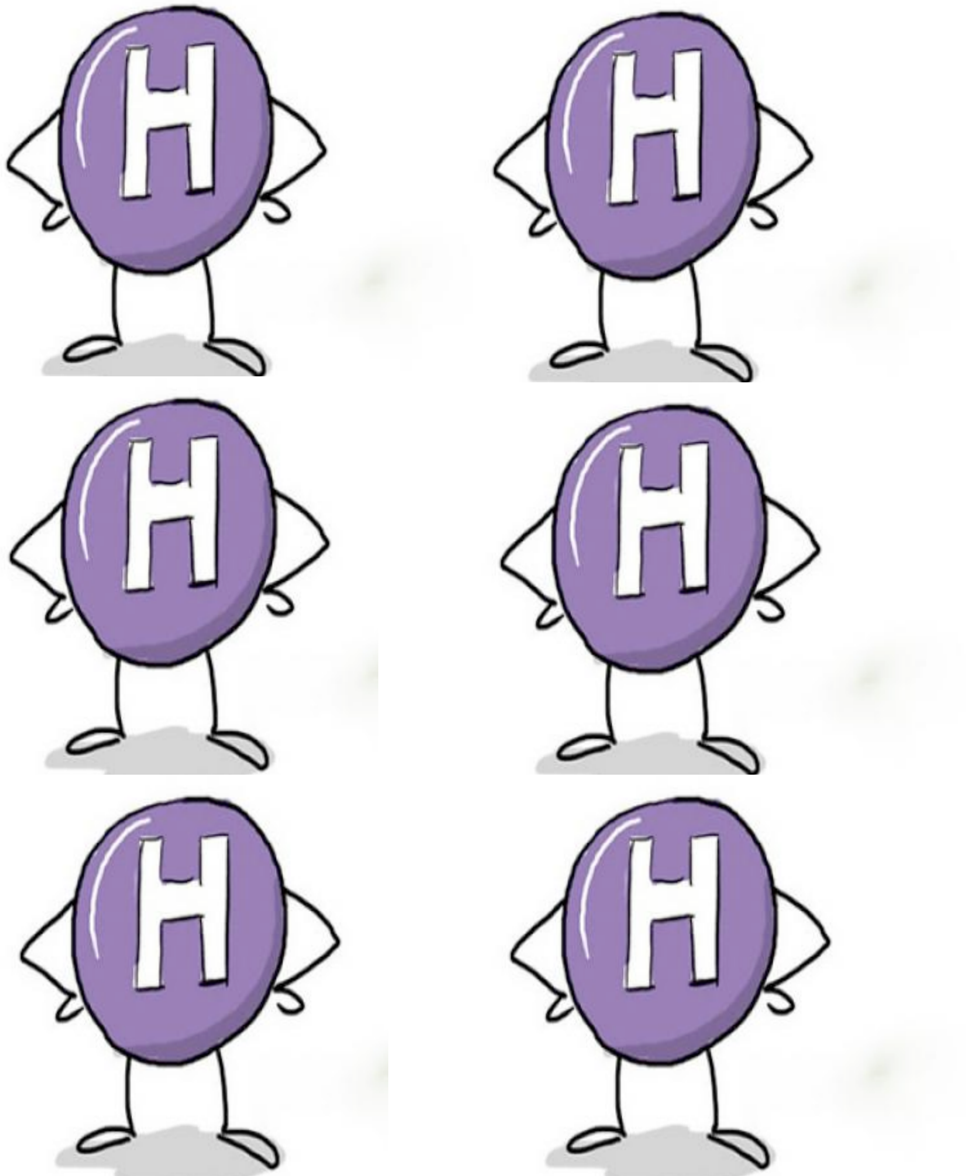
Replicating the Higgs Doublet



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Outline

- Extended Higgs sector—motivations and constraints
- A quick warmup—the 2HDM
- The NHDM and the charged Higgs basis
- HVV, HHV and $HHVV$ couplings
- Parameter counting
- Scalar couplings involving Goldstone bosons
- Scalar coupling sum rules
- Unitarity bounds
- Conditions for Higgs sector CP-conservation—a conjecture

This work is based on M.P. Bento, H.E. Haber, J.C. Romão and J.P. Silva, JHEP 11 (2017) 095 [arXiv:1708.09408].

Why not an extended Higgs sector?

- The fermion and gauge boson sectors of the Standard Model (SM) are not of minimal form ("Who ordered that?"). So, why should the spin-0 (scalar) sector be minimal?

- Extended Higgs sectors can provide a dark matter candidate.

- Extended Higgs sectors can modify the electroweak phase transition.

- Extended Higgs sectors can enhance vacuum stability.

- Models of new physics beyond the SM often require additional scalar Higgs states. E.g., two Higgs doublets are required in the minimal supersymmetric extension of the SM (MSSM).

Extended Higgs sectors are highly constrained

- The electroweak ρ parameter is very close to 1.
- One neutral Higgs scalar of the extended Higgs sector must be SM-like (and identified with the Higgs boson at 125 GeV).
- Higgs-mediated flavor-changing neutral currents (FCNCs) are suppressed.
- Charged Higgs exchange at tree level (e.g. in $\bar{B} \to D^{(*)}\tau^-\overline{\nu}_\tau$) and at one-loop (e.g. in $b \to s\gamma$) can significantly constrain the charged Higgs masses and the Yukawa couplings.
- At present, only one Higgs scalar has been observed.
- If the scale that governs the non-SM like Higgs bosons is close to the electroweak scale, is the naturalness problem of electroweak symmetry breaking exacerbated?

Constraints on extended Higgs sectors

In light of the observation that the electroweak ρ-parameter is very close to 1, it follows that a Higgs multiplet of weak-isospin T and hypercharge Y must satisfy,[1]

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1 \quad \Longleftrightarrow \quad (2T+1)^2 - 3Y^2 = 1\,,$$

independently of the Higgs vacuum expectation values (vevs). The simplest solutions are Higgs singlets $(T,Y) = (0,0)$ and hypercharge-one complex Higgs doublets $(T,Y) = (\frac{1}{2},1)$. In this talk, we shall neglect Higgs singlets and focus on models with N Higgs doublet fields (NHDM).

[1] Y is normalized such that the electric charge of the scalar field is $Q = T_3 + Y/2$.

Goals of the NHDM Study

- Identifying the physical parameters of the model

- Sum rules and unitarity bounds

- Dealing with CP-violation in the Higgs sector

The treatment of the Higgs-fermion interactions requires additional attention, in part due to the constraints of tree-level Higgs-mediated FCNCs. We defer this to another day, and focus in this talk on the bosonic sector of the NHDM.

A Warmup with the 2HDM

Consider the 2HDM with hypercharge-one, doublet scalar fields Φ_1 and Φ_2. After minimizing the scalar potential, $\langle\Phi_i^0\rangle = v_i/\sqrt{2}$ (for $i = 1, 2$), where $|v_1|^2+|v_2|^2 = (246~\mathrm{GeV})^2$ and $\tan\beta \equiv |v_2|/|v_1|$; the latter is basis-dependent and hence unphysical.

Introduce the Higgs basis fields,

$$H_1 = \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix} \equiv \frac{v_1^*\Phi_1 + v_2^*\Phi_2}{v}, \qquad H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} \equiv \frac{-v_2\Phi_1 + v_1\Phi_2}{v},$$

such that $\langle H_1^0\rangle = v/\sqrt{2}$ and $\langle H_2^0\rangle = 0$. The Higgs basis is uniquely defined up to an overall rephasing, $H_2 \to e^{i\chi}H_2$.

We can immediately identify the physical charged Higgs field, $H^+ \equiv H_2^+$, and the neutral and charged Goldstone fields, $G^0 = \sqrt{2}\,\mathrm{Im}\,H_1^0$ and $G^+ \equiv H_1^+$.

In the Higgs basis, the scalar potential is given by:

$$\begin{aligned}\mathcal{V} = {} & Y_1 H_1^\dagger H_1 + Y_2 H_2^\dagger H_2 + [Y_3 H_1^\dagger H_2 + \text{h.c.}] + \tfrac{1}{2} Z_1 (H_1^\dagger H_1)^2 \\ & + \tfrac{1}{2} Z_2 (H_2^\dagger H_2)^2 + Z_3 (H_1^\dagger H_1)(H_2^\dagger H_2) + Z_4 (H_1^\dagger H_2)(H_2^\dagger H_1) \\ & + \left\{ \tfrac{1}{2} Z_5 (H_1^\dagger H_2)^2 + \big[Z_6 (H_1^\dagger H_1) + Z_7 (H_2^\dagger H_2)\big] H_1^\dagger H_2 + \text{h.c.} \right\},\end{aligned}$$

where Y_1, Y_2 and $Z_{1,2,3,4}$ are real, whereas Y_3, $Z_{5,6,7}$ are potentially complex. After minimizing the scalar potential, $Y_1 = -\frac{1}{2} Z_1 v^2$ and $Y_3 = -\frac{1}{2} Z_6 v^2$.

Remarks:

1. Under the rephasing, $H_2 \to e^{i\chi} H_2$,

$$[Y_3, Z_6, Z_7] \to e^{-i\chi}[Y_3, Z_6, Z_7] \quad \text{and} \quad Z_5 \to e^{-2i\chi} Z_5\,.$$

2. Under the rephasing, $H_2 \to e^{i\chi} H_2$, the charged Higgs boson field is rephased, $H^\pm \to e^{\pm i\chi} H^\pm$

3. In the CP-conserving 2HDM, one can rephase the field H_2 such that all the parameters of the scalar potential are real.

Diagonalizing the neutral Higgs squared-mass matrix

The three physical neutral Higgs boson mass-eigenstates are determined by diagonalizing the 3×3 real symmetric squared-mass matrix in the Higgs basis,

$$\mathcal{M}^2 = v^2 \begin{pmatrix} Z_1 & \mathrm{Re}Z_6 & -\mathrm{Im}Z_6 \\ \mathrm{Re}Z_6 & \frac{1}{2}(Z_{345} + Y_2/v^2) & -\frac{1}{2}\mathrm{Im}Z_5 \\ -\mathrm{Im}Z_6 & -\frac{1}{2}\mathrm{Im}Z_5 & \frac{1}{2}(Z_{345} + Y_2/v^2) - \mathrm{Re}Z_5 \end{pmatrix},$$

where $Z_{345} \equiv Z_3 + Z_4 + \mathrm{Re}Z_5$. The diagonalization matrix is a 3×3 real orthogonal matrix that depends on three angles: θ_{12}, θ_{13} and θ_{23},

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{23} - c_{12}s_{13}s_{23} & -c_{12}s_{13}c_{23} + s_{12}s_{23} \\ s_{12}c_{13} & c_{12}c_{23} - s_{12}s_{13}s_{23} & -s_{12}s_{13}c_{23} - c_{12}s_{23} \\ s_{13} & c_{13}s_{23} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} \sqrt{2}\mathrm{Re}H_1^0 - v \\ \sqrt{2}\mathrm{Re}H_2^0 \\ \sqrt{2}\mathrm{Im}H_2^0 \end{pmatrix},$$

where the h_i are the mass-eigenstate neutral Higgs fields, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. We shall also denote the neutral Goldsotne boson, $h_0 \equiv G^0$.

That is, for $j = 0, 1, 2, 3$, the neutral Goldstone boson and the mass-eigenstate neutral Higgs fields are,

$$h_j = \frac{1}{\sqrt{2}}\left\{q_{j1}^* \left(H_1^0 - \frac{v}{\sqrt{2}}\right) + q_{j2}^* H_2^0 e^{i\theta_{23}} + \text{h.c.}\right\},$$

where

j	q_{j1}	q_{j2}
0	i	0
1	$c_{12}c_{13}$	$-s_{12} - ic_{12}s_{13}$
2	$s_{12}c_{13}$	$c_{12} - is_{12}s_{13}$
3	s_{13}	ic_{13}

Under the rephasing, $H_2 \to e^{i\chi} H_2$,

$$\theta_{12}\,,\ \theta_{13}\ \text{are invariant, and}\quad \theta_{23} \to \theta_{23} - \chi\,.$$

Thus the q_{jk} are linear combinations of the invariant angles θ_{12} and θ_{13}.

From the generic basis to the mass basis

In the generic basis,

$$\langle \Phi_i \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ \hat{v}_i \end{pmatrix},$$

where $v \simeq 246$ GeV and $\hat{v} = (\hat{v}_1, \hat{v}_2)$ is a complex vector of unit norm. A second unit vector, $\hat{w} \equiv \hat{v}_i^* \epsilon_{ij}$, can be defined that is orthogonal[2] to $\hat{v}$, where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$.

Note that the Higgs basis fields can be defined as $H_1 \equiv \hat{v}_i^* \Phi_i$ and $H_2 \equiv \hat{w}_i^* \Phi_i$. Under the rephasing $H_2 \to e^{i\chi} H_2$, the unit vector rephases, $\hat{w}_i \to e^{-i\chi} \hat{w}_i$.

With this notation, one can express the Φ_i (for $i = 1, 2$) in terms of mass-eigenstate fields,

$$\Phi_i = \begin{pmatrix} G^+ \hat{v}_i + H^+ \hat{w}_i \\ \dfrac{v}{\sqrt{2}} \hat{v}_i + \dfrac{1}{\sqrt{2}} \displaystyle\sum_{j=0}^{3} \left(q_{j1} \hat{v}_i + q_{j2} e^{-i\theta_{23}} \hat{w}_i \right) h_j \end{pmatrix}.$$

[2]Orthogonality s defined in terms of the complex dot product, $\sum_j \hat{v}_j^* \hat{w}_j = 0$.

Interaction of the Higgs bosons and gauge bosons

These interaction arise from the kinetic energy term of the scalars (after replacing the derivative with the gauge covariant derivative).

$$\mathscr{L}_{VVH} = \left(g m_W W_\mu^+ W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu \right) q_{k1} h_k + (e M_W A^\mu - g M_Z s_W^2 Z^\mu)(W_\mu^- G^+ + W_\mu^+ G^-),$$

$$\begin{aligned}
\mathscr{L}_{VVHH} &= \left[\tfrac{1}{4} g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{8c_W^2} Z_\mu Z^\mu \right] h_k h_k \\
&+ \left[\tfrac{1}{2} g^2 W_\mu^+ W^{\mu\,-} + e^2 A_\mu A^\mu + \frac{g^2}{c_W^2} \left(\tfrac{1}{2} - s_W^2 \right)^2 Z_\mu Z^\mu + \frac{2ge}{c_W} \left(\tfrac{1}{2} - s_W^2 \right) A_\mu Z^\mu \right] (G^+ G^- + H^+ H^-) \\
&+ \left\{ \left(\tfrac{1}{2} eg A^\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^+ \right) (q_{k1} G^- + q_{k2} e^{-i\theta_{23}} H^-) h_k + \text{h.c.} \right\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{VHH} &= -\frac{g}{4c_W} \epsilon_{jk\ell} q_{\ell 1} Z^\mu h_j \overleftrightarrow{\partial}_\mu h_k - \tfrac{1}{2} ig \left[W_\mu^+ (q_{k1} G^- \overleftrightarrow{\partial}^\mu h_k + q_{k2} e^{-i\theta_{23}} H^- \overleftrightarrow{\partial}^\mu h_k) + \text{h.c.} \right] \\
&\quad + \left[ieA^\mu + \frac{ig}{c_W} \left(\tfrac{1}{2} - s_W^2 \right) Z^\mu \right] (G^+ \overleftrightarrow{\partial}_\mu G^- + H^+ \overleftrightarrow{\partial}_\mu H^-) \,,
\end{aligned}$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$, and the sum over pairs of repeated indices $j, k = 0, 1, 2, 3$ is implied.

Invariant parameter counting

Total number of parameters for the 2HDM: Y_2, $Z_{1,2,3,4}$, complex $Z_{5,6,7}$ with one degree of freedom removed by rephasing, and $Y_{1,3}$ fixed by the scalar potential minimum conditions (in favor of the vev, v) yields 11 real parameters.

Not including the four masses (m_1, m_2, m_3 and $m_{H^\pm}$) and the vev v, this leaves 6 independent invariant parameters required to describe the bosonic interactions of the 2HDM.

Number of parameters governing the Higgs–gauge boson interactions: 2
2 invariant angles (θ_{12} and θ_{13}).

Additional invariant parameters arising via the Higgs self-interactions: 4

2HDM analysis based on H.E. Haber and D. O'Neil, Phys. Rev. D **74**, 015018 (2006) [Erratum: Phys. Rev. D **74**, 059905 (2006)] [hep-ph/0602242].

Analysis of the NHDM

Consider the NHDM with hypercharge-one, doublet scalar fields Φ_i, for $i = 1, 2, \ldots, N$. After minimizing the scalar potential, $\langle \Phi_i^0 \rangle = v_i/\sqrt{2}$, where $\sum_i |v_i|^2 = (246~\text{GeV})^2$.

Introduce the Higgs basis field,

$$H_1 = \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix} \equiv \frac{1}{v} \sum_{i=1}^{N} v_i^* \Phi_i \, ,$$

and the $N-1$ scalar fields, H_i $(i = 2, 3, \ldots, N)$ are all orthogonal to H_1, such that $\langle H_1^0 \rangle = v/\sqrt{2}$ and $\langle H_i^0 \rangle = 0$ for $i = 2, 3, \ldots, N$.

The Goldstone bosons reside in H_1, i.e. $G^+ = H_1^+$ and $G^0 = \sqrt{2}\,\text{Im}\,H_1^0$. But there is still too much freedom in defining the doublets orthogonal to H_1.

The charged Higgs basis

We shall perform a unitary transformation on the scalar doublet fields H_i ($i = 2, 3, \ldots, N$), such that the upper components of these doublets are charged mass eigenstates. We shall assume that these eigenstates are non-degenerate in mass (the degenerate case requires a separate consideration).

We call this choice of basis the *charged Higgs basis*. The corresponding Higgs doublet fields are denoted by H_i^C. Note that in this basis, the neutral components of the H_i^C are typically *not* neutral Higgs mass-eigenstates.

The charged Higgs basis is unique up to the separate rephasing of the $N-1$ Higgs doublet fields,

$$H_i^C \to e^{i\chi_i} H_i^C \,, \qquad \text{for } i = 2, 3, \ldots, N.$$

The phase of $H_1^C = H_1$ is fixed by the condition that the vev v is real and positive.

U and V matrices

In the generic basis,

$$\Phi_k = \begin{pmatrix} \varphi_k^+ \\ \frac{1}{\sqrt{2}}(v_k + \varphi_k^0) \end{pmatrix}, \qquad \text{for } k = 1, \ldots, N.$$

The charged Higgs basis is obtained via,

$$\Phi_j = \sum_{k=1}^{N} U_{jk} H_k^C \,,$$

with $H_1^C \equiv H_1$, U is an $N \times N$ unitary matrix such that $U_{j1} = v_j/v$, and

$$H_1^C = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \varphi_1^{C0}\right) \end{pmatrix}, \quad H_2^C = \begin{pmatrix} H_2^+ \\ \frac{1}{\sqrt{2}}\varphi_2^{C0} \end{pmatrix}, \ldots, \quad H_N^C = \begin{pmatrix} H_N^+ \\ \frac{1}{\sqrt{2}}\varphi_N^{C0} \end{pmatrix},$$

where $\varphi_1^{C0} \equiv H^0 + iG^0$ and the H_i^+ are charged Higgs mass eigenstates.

Diagonalizing the neutral Higgs squared-mass matrix

The neutral Higgs mass eigenstates will be denoted by h_k where $h_0 \equiv G^0$ and h_k for $k = 1, 2, \ldots, 2N-1$ are the physical neutral scalars. This is achieved by introducing the $N \times 2N$ matrix V,

$$\varphi_j^0 = \sum_{k=0}^{2N-1} V_{jk} h_k, \qquad \text{for } j = 1, 2, \ldots, N.$$

It is more convenient to first transform to the charged Higgs basis and then determine the neutral Higgs mass eigenstates,

$$\varphi_j^{C0} = \sum_{k=0}^{2N-1} B_{jk} h_k, \qquad \text{for } j = 1, 2, \ldots, N,$$

where $B = U^\dagger V$ defines an $N \times 2N$ matrix. The $2N \times 2N$ matrix,

$$\widetilde{B} \equiv \begin{pmatrix} \operatorname{Re} B \\ \operatorname{Im} B \end{pmatrix},$$

is real orthogonal. In particular, $\widetilde{B}^T$ is the matrix that converts the neutral scalars in the charged Higgs basis to the neutral Higgs mass eigenstates.

A and B matrices

$B = U^\dagger V$ is an $N \times 2N$ matrix. Under the separate rephasing of the $N-1$ Higgs doublet fields in the charged Higgs basis, $H_i^C \to e^{i\chi_i} H_i^C$ for $i = 2, 3, \ldots, N$, the matrix elements of B rephase as,

$$B_{jk} \to e^{i\chi_j} B_{jk}\,, \qquad \text{for } j = 2, 3, \ldots, N \text{ and } k = 0, 1, \ldots 2N-1,$$

and B_{1k} is invariant. Since $\widetilde{B}^T \widetilde{B} = \mathrm{Re}(B^\dagger B) = \mathbb{1}_{2N\times 2N}$, it follows that

$$B^\dagger B = \mathbb{1}_{2N\times 2N} + iA\,,$$

where $A \equiv \mathrm{Im}(B^\dagger B)$ is a real orthogonal antisymmetric $2N \times 2N$ matrix. The matrix A is invariant under the separate rephasing of the $N-1$ charged Higgs basis fields.

Finally, the following two properties are noteworthy,

$$B_{j0} = i\delta_{j1}\,, \qquad B_{1k} = -A_{0k} + i\delta_{0k}\,,$$

corresponding to identifying the neutral Goldstone boson eigenstate.

Interaction of the Higgs bosons and gauge bosons

$$\mathscr{L}_{VVH} = \left(g m_W W_\mu^+ W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu\right) A_{k0} h_k + (eM_W A^\mu - gM_Z s_W^2 Z^\mu)(W_\mu^- G^+ + W_\mu^+ G^-),$$

$$\mathscr{L}_{VVHH} = \left[\tfrac{1}{4} g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{8c_W^2} Z_\mu Z^\mu\right] h_k h_k$$
$$+ \left[\tfrac{1}{2} g^2 W_\mu^+ W^{\mu\,-} + e^2 A_\mu A^\mu + \frac{g^2}{c_W^2}\left(\tfrac{1}{2} - s_W^2\right)^2 Z_\mu Z^\mu + \frac{2ge}{c_W}\left(\tfrac{1}{2} - s_W^2\right) A_\mu Z^\mu\right] H_j^+ H_j^-$$
$$+ \left\{\left(\tfrac{1}{2} eg A^\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^+\right) B_{jk} H_j^- h_k + \text{h.c.}\right\},$$

$$\mathscr{L}_{VHH} = -\frac{g}{4c_W} A_{k\ell} Z^\mu h_k \overleftrightarrow{\partial}_\mu\, h_\ell - \tfrac{1}{2} g \left[i B_{jk} W_\mu^+ H_j^- \overleftrightarrow{\partial}^\mu\, h_k + \text{h.c.}\right]$$
$$+ \left[ieA^\mu + \frac{ig}{c_W}\left(\tfrac{1}{2} - s_W^2\right) Z^\mu\right] H_j^+ \overleftrightarrow{\partial}_\mu\, H_j^-\,,$$

where $H_1^+ \equiv G^+$, $j = 1, \ldots, N$, and $k, \ell = 0, 1, \ldots, 2N-1$. Note that A and the combination $B_{jk} H_j^-$ are invariant under the separate rephasing of the $N-1$ Higgs doublet fields in the charged Higgs basis, $H_i^C \to e^{i\chi_i} H_i^C$ for $i = 2, 3, \ldots, N$, as expected for the physical couplings.

Recovering the 2HDM results

The matrices U and V are,

$$U = \begin{pmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{pmatrix}, \qquad V_{ij} = q_{j1}\hat{v}_i + q_{j2}e^{-i\theta_{23}}\hat{w}_i \,.$$

This immediately yields the matrix B,

$$B = U^\dagger V = \begin{pmatrix} i & q_{11} & q_{21} & q_{31} \\ 0 & q_{12}e^{-i\theta_{23}} & q_{22}e^{-i\theta_{23}} & q_{32}e^{-i\theta_{23}} \end{pmatrix}.$$

Consider the 4×4 real orthogonal matrix $\widetilde{B} = \left(\begin{smallmatrix} \mathrm{Re}\, B \\ \mathrm{Im}\, B \end{smallmatrix}\right)$. Note that $\widetilde{B}_{31} = 1$ and all other elements appearing in the third row and first column vanish.[3]

Removing the third row and first column from $\widetilde{B}$, and taking the transpose of the resulting matrix, one recovers the 3×3 real orthogonal matrix that diagonalizes the neutral Higgs squared-mass matrix.

[3]The third row and first column of $\widetilde{B}$ are associated with the neutral Goldstone boson eigenstate.

Finally, the matrix A is invariant with respect to the rephasing of the Higgs field H_2,

$$A = \mathrm{Im}(B^\dagger B) = \begin{pmatrix} 0 & -q_{11} & -q_{21} & -q_{31} \\ q_{11} & 0 & \mathrm{Im}(q_{12}^* q_{22}) & \mathrm{Im}(q_{12}^* q_{32}) \\ q_{21} & -\mathrm{Im}(q_{12}^* q_{22}) & 0 & \mathrm{Im}(q_{22}^* q_{32}) \\ q_{31} & -\mathrm{Im}(q_{12}^* q_{32}) & -\mathrm{Im}(q_{22}^* q_{32}) & 0 \end{pmatrix},$$

Using the values of the q_{jk} yields,

$$A = \begin{pmatrix} 0 & -c_{12}c_{13} & -s_{12}c_{13} & -s_{13} \\ c_{12}c_{13} & 0 & s_{13} & -s_{12}c_{13} \\ s_{12}c_{13} & -s_{13} & 0 & c_{12}c_{13} \\ s_{13} & s_{12}c_{13} & -c_{12}c_{13} & 0 \end{pmatrix}.$$

which is the most general 4×4 real orthogonal antisymmetric matrix.

Invariant parameter counting

Instead of B, consider the $2N \times 2N$ real orthogonal matrix $\widetilde{B} = \left(\begin{smallmatrix} \mathrm{Re}\, B \\ \mathrm{Im}\, B \end{smallmatrix} \right)$. After removing the Goldstone boson eigenstate, one is left with a $2N-1 \times 2N-1$ real orthogonal matrix.

$A = \mathrm{Im}(B^\dagger B)$ is a real orthogonal antisymmetric $2N \times 2N$ matrix.

matrix	parameters	unphysical phases	physical parameters
B	$(N-1)(2N-1)$	$N-1$	$2(N-1)^2$
A	$N(N-1)$	0	$N(N-1)$

Note that since A is determined by B, the number of parameters governing A cannot be larger than the parameters that govern B. Indeed,

$$2(N-1)^2 \geq N(N-1),$$

with equality when $N = 1$ or 2, and inequality for $N > 2$.

To determine the total number of real parameters that govern the NHDM, start with the Higgs scalar potential in the charged Higgs basis,

$$\mathcal{V} = Y_{ij}(H_i^{C\dagger} H_j^C) + \tfrac{1}{2} Z_{ij,k\ell}(H_i^{C\dagger} H_j^C)(H_k^{C\dagger} H_\ell^C).$$

	magnitudes	phases	constraints	parameters
Y	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$	$2(N-1)$	N^2-2N+2
Z	$\frac{1}{4}N^2(N^2+3)$	$\frac{1}{4}N^2(N^2-1)$	$N-1$	$\frac{1}{2}(N^4+N^2-2N+2)$
Y and Z	$\frac{1}{4}(N^4+5N^2+2N)$	$\frac{1}{4}(N^4+N^2-2N)$	$3N-3$	$\frac{1}{2}(N^4+3N^2-6N+6)$

We still have to impose the scalar potential minimum conditions,

$$Y_{1j} + \tfrac{1}{2}v^2 Z_{1j,11} = 0\,, \qquad \text{for } j = 1, 2, \ldots, N.$$

We can trade in Y_{11} (which is real) for v^2. The $N-1$ complex quantities Y_{1j} for $j = 2, 3, \ldots, N$ can be re-expressed in terms of the $Z_{1j;11}$, which yields $2(N-1)$ real constraints. Finally, the complex elements of $Z_{ij;k\ell}$ can be rephased by $H_\ell^C \to e^{i\chi_\ell} H_\ell^C$ for $\ell = 2, 3, \ldots, N$, thereby removing $N-1$ phases.

Final scorecard

number of parameters governing the Higgs–vector boson couplings: $2(N-1)^2$

number of charged Higgs masses: $N-1$

number of neutral Higgs masses: $2N-1$

Higgs vev (v): 1

additional parameters associated with the Higgs self-couplings:

$$\tfrac{1}{2}(N-1)(N^3+N^2-4)$$

Total number of parameters governing the NHDM:

$$\tfrac{1}{2}(N^4+3N^2-6N+6)$$

.

Scalar couplings involving the Goldstone boson

In general the scalar self-interactions involve additional (pseudo-)invariant quantities beyond A and B. Remarkably, certain scalar couplings involving the Goldstone bosons again depend only on A and B (and the Higgs masses).

coupling	$-iv\times$Feynman rule
$G^0 h_i h_j \ (i \neq j)$	$(m_j^2 - m_i^2) A_{ij}$
$G^+ H_j^- h_k$	$(m^2_{H_j^\pm} - m_k^2) B_{jk}$
$G^- H_j^+ h_k$	$(m^2_{H_j^\pm} - m_k^2) B^\dagger_{jk}$
$G^+ G^- h_k$	$m_k^2 A_{0k}$
$G^0 G^0 h_k$	$m_k^2 A_{0k}$
$G^+ G^- h_k$	$m_k^2 A_{0k}$

The couplings, $G^0 H_i^+ H_j^-$, $G^0 h_i h_i$, $G^0 G^\pm H_j^\mp$, $G^0 G^0 G^0$, and $G^0 G^+ G^-$, vanish.

The quartic scalar couplings involving two or more Goldstone boson fields depend only on A, B and Y (and the Higgs masses). For example,

coupling	$-iv^2 \times$Feynman rule
$G^+G^-G^+G^-$	$2\sum_{k=1}^{2N-1} m_k^2 (A_{0k})^2$
$G^+G^-G^0G^0$	$\sum_{k=1}^{2N-1} m_k^2 (A_{0k})^2$
$G^0G^0G^0G^0$	$3\sum_{k=1}^{2N-1} m_k^2 (A_{0k})^2$
$G^+G^+H_i^-H_j^-$	$2\sum_{k=1}^{2N-1} m_k^2 B_{ik}B_{jk}$
$G^-G^-H_i^+H_j^+$	$2\sum_{k=1}^{2N-1} m_k^2 B_{ik}^*B_{jk}^*$
$G^+G^-H_i^+H_j^-$	$\sum_{k=1}^{2N-1} m_k^2 B_{ik}^*B_{jk} - 2Y_{ji}$
$G^+G^-h_kh_\ell$	$2\left[(B^\dagger D_\pm^2 B)_{k\ell} - (B^\dagger Y B)_{k\ell}\right]$
$G^0G^0H_i^+H_j^-$	$2\left[(D_\pm^2)_{ji} - Y_{ji}\right]$
$G^0G^0h_kh_\ell$	$2\left[(AD_0^2A)_{k\ell} - (B^\dagger Y B)_{k\ell}\right]$

where $D_\pm^2$ and D_0^2 are the diagonal charged and neutral Higgs squared-mass matrices, respectively. Using $H_1^\pm = G^\pm$ and $h_0 = G^0$, we can check for consistency after noting that $Y_{j1} = \frac{1}{2}\sum_{k=1}^{2N-1} m_k^2 A_{0k} B_{jk}\,.$

Scalar coupling sum rules

Given the NHDM Lagrangian, one can read off a variety of useful scalar coupling sum rules. Ultimately, these sum rules are a consequence of unitarity. Perhaps the most well known sum rule of this type is

$$\sum_k [h_k VV]^2 = 1\,,$$

where $VV = W^+W^-$ or ZZ, and $[h_k VV]$ is our notation for the coefficient of the corresponding term that appears in the Lagrangian up to an overall normalization (taken to be that of the Standard Model, if it exists). Other sum rules of interest are (recall that $H_1^\pm = G^\pm$ and $h_0 = G^0$):

$$\sum_{j=1}^{N} \left|[h_k W^\mp H_j^\pm]\right|^2 = 1\,, \qquad \sum_{k=0}^{2N-1} [Zh_k h_\ell]^2 = 1\,,$$

$$\sum_{k=0}^{2N-1} [W^+ H_j^- h_k][Zh_k h_\ell] = -i[W^+ H_j^- h_\ell]\,.$$

A somewhat related sum rule is,

$$[h_\ell VV]^2 + \sum_{k=1}^{2N-1} [Zh_k h_\ell] = 1\,,$$

which is valid for $\ell = 0, 1, 2, \ldots, 2N-1$.

We have also uncovered a number of interesting sum rules involving quartic scalar couplings,

$$\sum_{j=1}^{N} [G^+G^-H_i^+H_i^-] = \mathrm{Tr}(D_0^2) - 2\mathrm{Tr}(Y)\,,$$

$$\sum_{k=1}^{2N-1} [G^+G^-h_k h_k] = 2\sum_{j=1}^{N} [G^0G^0H_j^+H_j^-] = 2\,\mathrm{Tr}(D_\pm^2) - 2\mathrm{Tr}(Y)\,.$$

Unitarity bounds for specific processes

Focusing on specific processes of the NHDM will yield necessary (but not sufficient) conditions that the tree-level unitarity bounds are satisfied. We use the equivalence theorem to replace the W and Z with the respective Goldstone bosons $G^\pm$ and G^0. After computing the relevant scattering matrix element $\mathcal{M}$, we shall impose the condition, $|\mathrm{Re}\,\mathcal{M}| \leq 8\pi$.

Example 1: $W^+W^- \to W^+W^-$

Here we compute the matrix element for $G^+G^- \to G^+G^-$. The leading order contribution is due to the corresponding quartic coupling,

$$\mathcal{M}\left(G^+G^- \to G^+G^-\right) = -\frac{2}{v^2}\sum_{k=1}^{2N-1} m_k^2 \left[A_{0k}\right]^2 = -\frac{2}{v^2}\sum_{k=1}^{2N-1} m_k^2 \left[h_k VV\right]^2 .$$

which the yields the bound,

$$\left|\sum_{k=1}^{2N-1} m_k^2 \left[h_k VV\right]^2\right| \leq (872\,\mathrm{GeV})^2 .$$

Example 2: $ZH_j^+ \to ZH_j^+$

Here we compute the matrix element for $G^0 H_j^+ \to G^0 H_j^+$– (for $j \geq 2$). The leading order contribution is due to the corresponding quartic coupling,

$$\mathcal{M}(G^0 H_j^+ \to G^0 H_j^+) = -\frac{2}{v^2}\big(m^2_{H_j^\pm} - Y_{jj}\big)\,,$$

which yields a bound on the NHDM parameters, $\left|m^2_{H_j^\pm} - Y_{jj}\right| \leq 4\pi v^2$.

Example 3: $W^+ H_j^- \to W^+ H_j^-$

Following the previous example,

$$\mathcal{M}\left(G^+ H_j^- \to G^+ H_j^-\right) = -\frac{1}{v^2}\left[(BD_0^2 B^\dagger)_{jj} - 2Y_{jj}\right]\,,$$

The resulting bound, $\left|(BD_0^2 B^\dagger)_{jj} - 2Y_{jj}\right| \leq 8\pi v^2$, can be rewritten in terms of one of the VHH couplings,

$$\left|\sum_{k=1}^{2N-1} m_k^2\,[W^+ H_j^- h_k]\,[W^- H_j^+ h_k] - 2Y_{jj}\right| \leq 8\pi v^2\,.$$

This provides an explicit constraint on the cubic couplings shown above.

Higgs sector CP-violation—a conjecture

If the charged Higgs bosons are non-degenerate in mass, then the Higgs scalar potential and the vacuum conserve CP if and only if one can find an appropriate rephasing of the doublet Higgs fields in the charged Higgs basis, $H_\ell^C \to e^{i\chi_\ell} H_\ell^C$ for $\ell = 2, 3, \ldots, N$, such that all the Y_{ij} and $Z_{ij;k\ell}$ are real.

An interesting "counterexample" due to I. Ivanov and J.P. SIlva is the 3HDM with order-4 CP-symmetry, with λ_8 and λ_9 complex.

$$
\begin{aligned}
V = & - m_{11}^2(\Phi_1^\dagger\Phi_1) - m_{22}^2(\Phi_2^\dagger\Phi_2 + \Phi_3^\dagger\Phi_3) + \lambda_1(\Phi_1^\dagger\Phi_1)^2 + \lambda_2\left[(\Phi_2^\dagger\Phi_2)^2 + (\Phi_3^\dagger\Phi_3)^2\right] \\
& + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2 + \Phi_3^\dagger\Phi_3) + \lambda_3'(\Phi_2^\dagger\Phi_2)(\Phi_3^\dagger\Phi_3) + \lambda_4\left[(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + (\Phi_1^\dagger\Phi_3)(\Phi_3^\dagger\Phi_1)\right] \\
& + \lambda_4'(\Phi_2^\dagger\Phi_3)(\Phi_3^\dagger\Phi_2) + \left[\lambda_5(\Phi_3^\dagger\Phi_1)(\Phi_2^\dagger\Phi_1) + \frac{\lambda_6}{2}\left[(\Phi_2^\dagger\Phi_1)^2 - (\Phi_1^\dagger\Phi_3)^2\right]\right. \\
& \left. + \lambda_8(\Phi_2^\dagger\Phi_3)^2 + \lambda_9(\Phi_2^\dagger\Phi_3)(\Phi_2^\dagger\Phi_2 - \Phi_3^\dagger\Phi_3) + \text{h.c.}\right],
\end{aligned}
$$

Although this 3HDM exhibits a generalized CP symmetry, there exists no basis in which all the scalar potential parameters are real. However, this model does exhibit mass-degenerate charged Higgs bosons, and thus is not in contradiction to the conjecture stated above.