Thermal regularization of t-channel singularities of scattering processes

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based on

B. Grządkowski, M. Iglicki, S. Mrówczyński, Nucl.Phys.B 984 (2022) 115967 ⟵ scalars :)

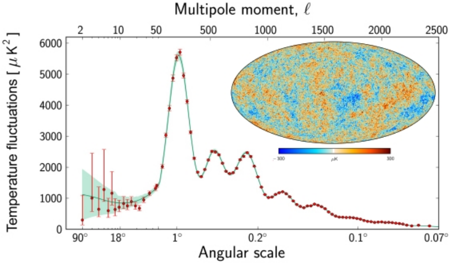
M. Iglicki, JHEP 06 (2023) 006 ⟵ general case

Scalars 2023

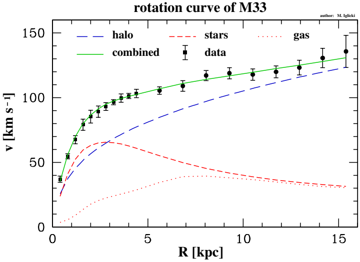
Warsaw, 15 September 2023

Motivation: dark matter

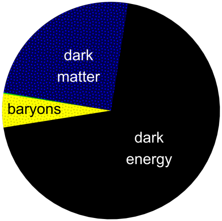
- overwhelming evidence
- most probably: BSM particles
- possibly multi-component



source: http://sci.esa.int/planck, https://wiki.cosmos.esa.int/planck-legacy-archive



source of the data: astro-ph/9909252



source: https://apod.nasa.gov/apod

- how much dark matter (or anything else) is there?

$$\dot{n}_x + \overbrace{3\,H\,n_x}^{\text{Universal expansion}} = \overbrace{- \sum_{2\to F} C^{x}_{ij\to f_1\ldots f_F}\, \langle \sigma v \rangle_{ij\to f_1\ldots f_F} \left[n_i n_j - \bar{n}_i \bar{n}_j \frac{n_{f_1}\ldots n_{f_F}}{\bar{n}_{f_1}\ldots \bar{n}_{f_F}} \right] + \text{decay terms}}^{\text{particle physics}}$$

Hubble parameter

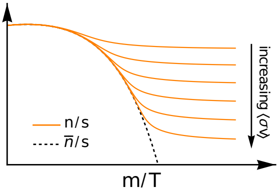
number density

comb. factor

thermally avgd. cross section

equilibrium value of n

$\Downarrow$



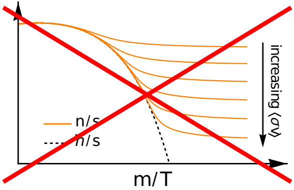
(references and more details → backup slides)

Motivation: the Boltzmann equation

- how much dark matter (or anything else) is there?

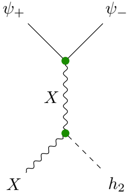
$$\dot{n}_x + \overbrace{3\,H\,n_x}^{\text{Universal expansion}} = \overbrace{-\sum_{2\to F} C^x_{ij\to f_1\dots f_F} \underset{\infty}{\cancel{\langle\sigma v\rangle}}_{ij\to f_1\dots f_F}\left[n_i n_j - \bar{n}_i\bar{n}_j \frac{n_{f_1}\dots n_{f_F}}{\bar{n}_{f_1}\dots\bar{n}_{f_F}}\right]}^{\text{particle physics}} + \text{decay terms}$$

$\Downarrow$



- problem: in multi-component DM scenarios $\langle\sigma v\rangle$ is sometimes singular!

example:



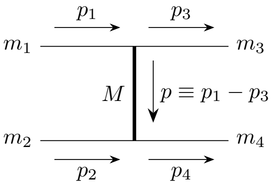
[GeV]				
m_X	m_{ψ_+}	m_{ψ_-}	m_{h_2}	$\langle\sigma v\rangle$
90	130	30	160	**finite**
70	130	30	160	**singular**
70	130	80	160	**finite**

~~numerical errors?~~
physics, actually!

(VFDM model: A. Ahmed et al., Eur.Phys.J.C 78 (2018) 11, 905)

← Higgs-portal model, scalars again :)

t-channel singularity: where the infinities come from



$$\mathcal{M} \sim \frac{1}{t - M^2}\,, \qquad t \equiv p^2$$

$$t = M^2 \quad \Rightarrow \quad \text{singular matrix element}$$
$$\Rightarrow \quad \text{infinite cross section}$$

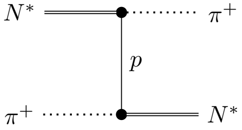
- IR regularization not applicable if $M > 0$
- Dyson resummation not helpful if $\Gamma = 0$

$\Rightarrow$ genuine singularity if the mediator is massive and stable

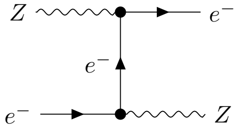
t-channel singularity: examples

noticed for the 1st time:
hadronic scattering

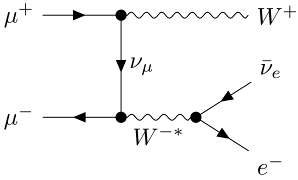
R. F. Peierls, Phys. Rev. Lett. 6 (1961) 641–643



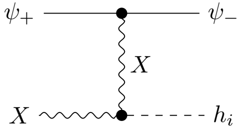
SM: weak Compton scattering



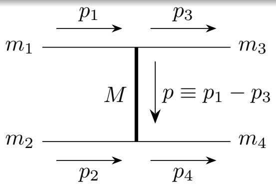
SM: muon colliders



BSM: dark matter in the early Universe



$2 \leftrightarrow 2$ process: when does the t-channel singularity occur?



$$\mathcal{M} \sim \frac{1}{t - M^2}$$

$$t \equiv p^2 = (p_1 - p_3)^2$$

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$

- singularity $\Leftrightarrow$ $\boxed{t = M^2}$ (massive, stable mediator required)

- cross section

$$\sigma(s) \supset \int_{t_{\text{min}}(s)}^{t_{\text{max}}(s)} \frac{dt}{(t - M^2)^2}$$

- thermally averaged cross section

$$\langle \sigma v \rangle (T) \supset \int \sigma(s)\, f(E_1, E_2, T)\, ds$$

- singularity condition

$$t_{\text{min}}(s) < M^2 < t_{\text{max}}(s)$$

- from kinematics

$$t_{\text{min}}(s) = \underbrace{m_1^2 + m_3^2 - 2E_1E_3 - 2|\mathbf{p}_1||\mathbf{p}_3|}_{\text{function of } s \text{ and masses}} \qquad t_{\text{max}}(s) = \underbrace{m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|}_{\text{function of } s \text{ and masses}}$$

$2 \leftrightarrow 2$ process: when does the t-channel singularity occur?

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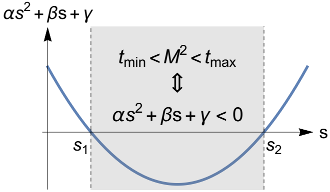
- in terms of the CM energy ($\sqrt{s}$)

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

$$\Updownarrow$$

$$\boxed{s_1 < s < s_2}$$

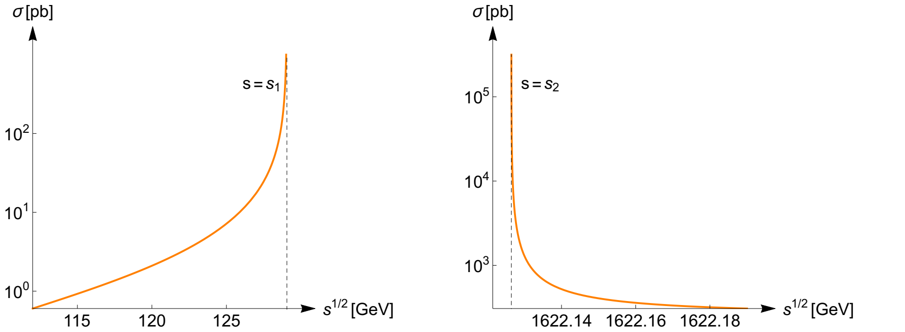


$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

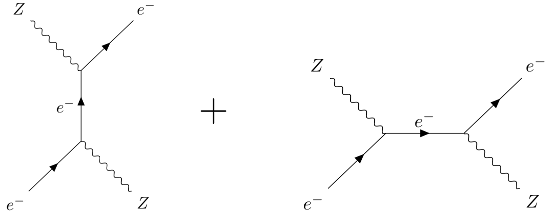
α, β, γ – functions of m_1, m_2, m_3, m_4, M

example: weak Compton scattering

$$s_1 = 2\,m_Z^2 + m_e^2 \xrightarrow{m_e \to 0} 2\,m_Z^2 \qquad\qquad s_2 = (m_Z^2 - m_e^2)^2/m_e^2 \xrightarrow{m_e \to 0} \infty$$



$$Ze^- \to Ze^- =$$



- singularity condition

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

$$t_{\min}(s) = \underbrace{m_1^2 + m_3^2 - 2E_1E_3 - 2|\mathbf{p}_1||\mathbf{p}_3|}_{\text{function of } s \text{ and masses}} \qquad t_{\max}(s) = \underbrace{m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|}_{\text{function of } s \text{ and masses}}$$

- in terms of the CM energy ($\sqrt{s}$)

$$t_{\min}(s) < M^2 < t_{\max}(s)$$

$$\Updownarrow$$

$$\boxed{s_1 < s < s_2}$$

- thermally averaged cross section

$$\langle \sigma v \rangle(T) \supset \int_{s_{\min}}^{\infty} \sigma(s)\, f(E_1, E_2, T)\, ds$$

functions of masses

- conclusion: $\langle \sigma v \rangle$ is singular if

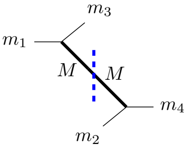
$$\boxed{s_2 > s_{\min} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}}$$

$\langle \sigma v \rangle$ is singular if

$$\boxed{\underbrace{s_2 > s_{\min}}_{\text{functions of masses}} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}}$$

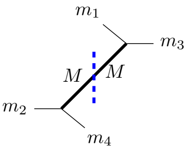
$\Updownarrow$ (some algebra)

$$m_1 > M + m_3 \text{ and } m_4 > M + m_2$$



or

$$m_2 > M + m_4 \text{ and } m_3 > M + m_1$$



◇ Coleman-Norton theorem

S. Coleman & R. E. Norton, Nuovo Cim 38, 438–442 (1965)

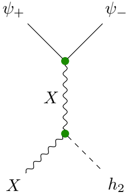
"It is shown that a Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time"

note: one of the external states decays
so it cannot be an asymptotic state
$\Rightarrow$ the usual approach to σ invalid

Coming back to our example. . .

- condition for the singularity to occur:

$$m_{\psi_+} > m_X + m_{\psi_-} \qquad \text{and} \qquad m_{h_2} > 2\, m_X$$



[GeV]						
m_X	m_{ψ_+}	m_{ψ_-}	m_{h_2}	$\langle\sigma v\rangle$	$\psi_+ \to \psi_- X$	$h_2 \to X X$
90	130	30	160	**finite**	**allowed**	forbidden
70	130	30	160	**singular**	**allowed**	**allowed**
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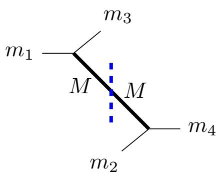
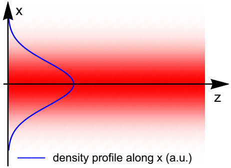


- complex mass of unstable particles

  I. Ginzburg, Nucl.Phys.B Proc.Suppl. 51 (1996) 85-89

  - finite lifetime of the particle should affect the wavefunction
  - $p_k \to \tilde{p}_k \equiv p_k + i\, p'_k(\Gamma_k)$
  - problem: $(\widetilde{p}_1 - \widetilde{p}_3)^2 \neq (\widetilde{p}_4 - \widetilde{p}_2)^2 \Rightarrow$ lack of symmetry



- finite beam width

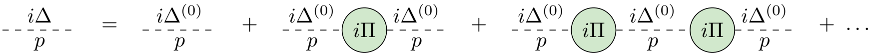
  G. L. Kotkin et al., Yad. Fiz. 42 (1982) 692
  G. L. Kotkin et al., Int. Journ. Mod. Phys. A 7 (1992) 4707
  K. Melnikov & V. G. Serbo, Nucl.Phys. B483 (1997) 67
  C. Dams & R. Kleiss, Eur.Phys.J.C29 (2003) 11
  C. Dams & R. Kleiss, Eur.Phys.J. C36 (2004) 177
  also related: D. Karamitros, A. Pilaftsis, Phys.Rev.D 108 (2023) 3, 036007

  - $n(x,y) \sim e^{-\frac{x^2+y^2}{2\,a^2}} \neq 1$
    $\Rightarrow$ momentum uncertainty
  - final results proportional to the width: $\int dt|\mathcal{M}|^2 \sim a$
  - works for colliders, but inapplicable in the cosmological context

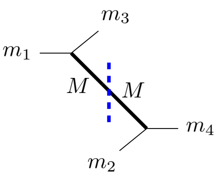
- Dyson resummation

  any textbook :)

  - $[p^2 - M^2]^{-1} \to [p^2 - M^2 + \Pi]^{-1}$
  - $\Im\Pi \xrightarrow{p^2 \to M^2}$ [decay width] $\Rightarrow$ no regularization for a stable mediator



Known approaches to the problem



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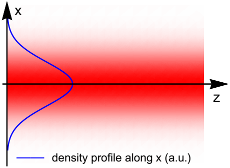

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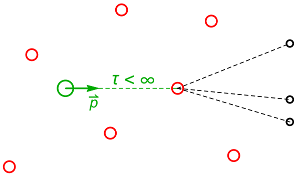
  - $[p^2 - M^2]^{-1} \longrightarrow [p^2 - M^2 + \Pi]^{-1}$
  - $\Im\Pi \xrightarrow{p^2 \to M^2}$ [decay width] $\Rightarrow$ no regularization for a $\overset{?}{\text{stable}}$ mediator

$$\frac{i\Delta}{p} = \frac{i\Delta^{(0)}}{p} + \frac{i\Delta^{(0)}}{p}\, i\Pi\, \frac{i\Delta^{(0)}}{p} + \frac{i\Delta^{(0)}}{p}\, i\Pi\, \frac{i\Delta^{(0)}}{p}\, i\Pi\, \frac{i\Delta^{(0)}}{p} + \ldots$$

Idea

- early Universe = hot gas
- every particle interacts with a thermal medium
- the mean life time cannot be infinite $\Rightarrow$ effective width
- QFT in a thermal medium: Keldysh-Schwinger formalism

vacuum



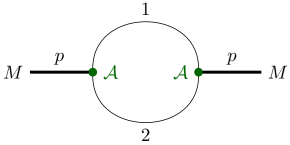
early Universe

note:

- similar results for the effective width by H.A. Weldon, Phys. Rev. D 28 (1983) 2007
- what is novel, is the application of that idea to early-Universe DM physics

vacuum

- one-loop self energy



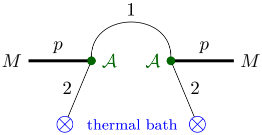
$$i\Pi(x,y) = i\Delta_1(x,y)\,\mathcal{A}(y)\,i\Delta_2(y,x)\,\mathcal{A}(x)$$

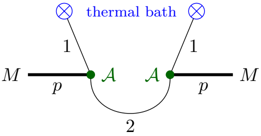
optical thm. $\Rightarrow$ $\Im\Pi = 0$ for stable M

note: hereafter, m_1 and m_2 are masses of the loop states

early Universe

- non-zero imaginary part of the self-energy appears as a result of interactions with the thermal medium





condition: $m_1 > m_2 + M$ or $m_2 > m_1 + M$

mediator $+$ thermal bath particle

$\updownarrow$

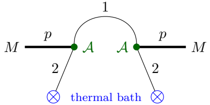
unstable intermediate state

$$\Pi^+(p,T) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left[\Delta_1^+(k+p)\,\mathcal{A}\,\Delta_2^{\mathsf{sym}}(k,T)\,\mathcal{A} + \Delta_1^{\mathsf{sym}}(k,T)\,\mathcal{A}\,\Delta_2^-(k-p)\,\mathcal{A}\right]$$

(more details $\longrightarrow$ backup slides)

- after tedious calculations...

  (assumption: $m_1 > m_2 + M$)



$$\Sigma(|\mathbf{p}|,T) \equiv \Im\Pi^{+}(|\mathbf{p}|,T)$$
$$= \frac{1}{16\pi}\,\frac{X_0}{\beta|\mathbf{p}|}\,\ln\left[1+\eta_2\,\frac{e^{-\beta(b-a)}e^{\beta E_p}\left(1-e^{-2\beta a}\right)\left(1-\eta_1\eta_2\,e^{-\beta E_p}\right)}{\left(1+\eta_1\,e^{-\beta(b-a)}\right)\left(1+\eta_2\,e^{-\beta(b+a)}e^{\beta E_p}\right)}\right]$$

$$a \equiv \frac{\lambda(m_1^2,m_2^2,M^2)^{1/2}}{2\,M^2}\,|\mathbf{p}|\;, \qquad b \equiv \frac{m_1^2-m_2^2+M^2}{2\,M^2}\,E_p\;, \qquad E_p \equiv \sqrt{\mathbf{p}^{\,2}+M^2}$$

$$\lambda(m_1^2,m_2^2,M^2) \equiv [m_1^2-(m_2+M)^2]\,[m_1^2-(m_2-M)^2]$$

$\eta_i \equiv +1$ for fermions, -1 for bosons ,
$$X_0 = \eta_2\,|\mathcal{M}|^2_{\text{dec}} \times \begin{cases} 1 & \text{scalar} \\ 1/2 & \text{fermion} \\ 1/3 & \text{vector} \end{cases}$$

- regularization:

effective width

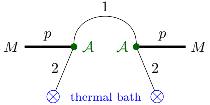
$$\Gamma_{\text{eff}}(|\mathbf{p}|,T) \equiv M^{-1}\Sigma(|\mathbf{p}|,T)$$

$\Rightarrow$

resummed propagator

$$\frac{1}{(t-M^2)^2} \;\to\; \frac{1}{(t-M^2)^2+M^2\Gamma_{\text{eff}}(|\mathbf{p}|,T)^2}$$

- after tedious calculations...
  (assumption: $m_1 > m_2 + M$)



$\Sigma(|\mathbf{p}|, T) \equiv \Im\Pi^+(|\mathbf{p}|, T)$ **temperature dependence,** $\Sigma \xrightarrow{T\to 0} 0$

$$= \frac{1}{16\pi}\,\frac{X_0}{\beta|\mathbf{p}|}\,\ln\left[1 + \eta_2\,\frac{e^{-\beta(b-a)}e^{\beta E_p}\boxed{\left(1 - e^{-2\beta a}\right)}\left(1 - \eta_1\eta_2\,e^{-\beta E_p}\right)}{\left(1 + \eta_1\,e^{-\beta(b-a)}\right)\left(1 + \eta_2\,e^{-\beta(b+a)}e^{\beta E_p}\right)}\right]$$

vanishes for $m_1 \to m_2 + M$

$$\boxed{a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2\,M^2}\,|\mathbf{p}|\,,} \qquad b \equiv \frac{m_1^2 - m_2^2 + M^2}{2\,M^2}\,E_p\,, \qquad E_p \equiv \sqrt{\mathbf{p}^2 + M^2}$$

$$\lambda(m_1^2, m_2^2, M^2) \equiv [m_1^2 - (m_2 + M)^2]\,[m_1^2 - (m_2 - M)^2]$$

$$\eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons}\,, \qquad X_0 = \eta_2\,|\mathcal{M}|^2_{\text{dec}} \times \begin{cases} 1 & \text{scalar} \\ 1/2 & \text{fermion} \\ 1/3 & \text{vector} \end{cases}$$

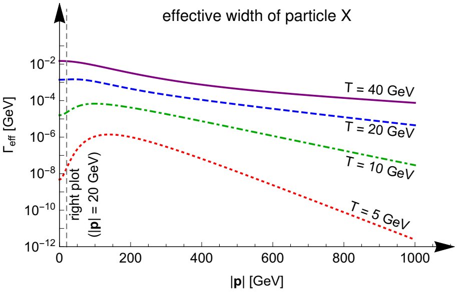
spin dependence

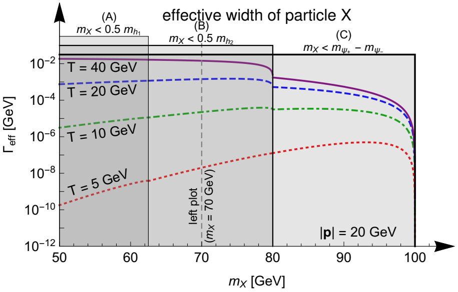
- regularization:

effective width $\Rightarrow$ resummed propagator

$$\Gamma_{\text{eff}}(|\mathbf{p}|, T) \equiv M^{-1}\Sigma(|\mathbf{p}|, T) \quad \Rightarrow \quad \frac{1}{(t-M^2)^2} \to \frac{1}{(t-M^2)^2 + M^2\Gamma_{\text{eff}}(|\mathbf{p}|,T)^2}$$

Numerical example: effective width



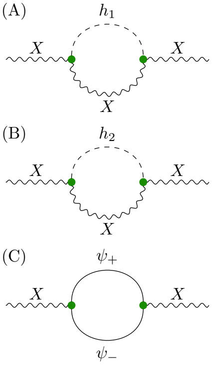


$m_X = 70$ GeV (upper plot)

$m_{\psi_+} = 130$ GeV $\quad m_{\psi_-} = 30$ GeV

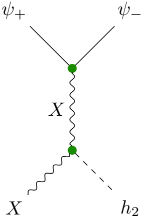
$m_{h_1} = 125$ GeV $\quad m_{h_2} = 160$ GeV

$g_x = 0.1 \quad \sin\alpha = 0.1$



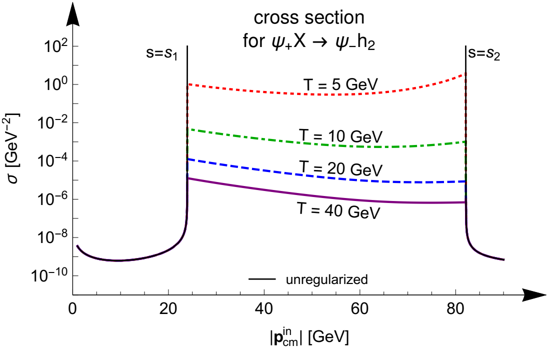
VFDM model: A. Ahmed et al., Eur.Phys.J.C 78 (2018) 11, 905

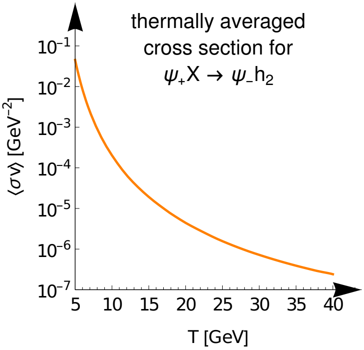
Numerical example: thermally averaged cross section



$$\langle\sigma v\rangle_{12\to34}(T) = \int d\Phi_1\, d\Phi_2\, f(E_1,E_2,T)$$
$$\times \int d\Phi_3\, d\Phi_4\, |\mathcal{M}|^2_{\text{dec}}\, \frac{(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)}{(t-M^2)^2 + M^2\,\Gamma_{\text{eff}}(|\mathbf{p}|,T)^2}$$

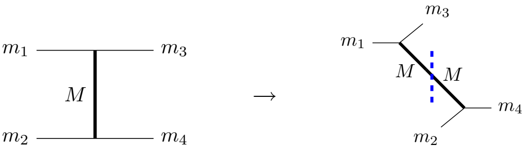
$$d\Phi_i \equiv \frac{d^3p_i}{(2\pi)^3\, 2E_i}$$ – phase-space element



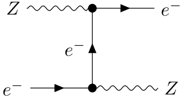


note: results still huge $\Rightarrow$ possible impact on DM evolution //TODO

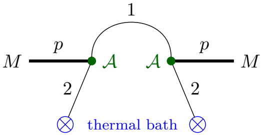
- t-channel singularity of $\langle \sigma v \rangle$ occurs if
  - the process can be seen as a sequence of decay and fusion processes



  - the mediator is massive and stable



- the singularity is present both in SM and BSM physics
- previously proposed approaches are either unsatisfactory or inapplicable

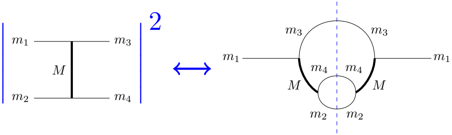


- interactions with the medium result in a non-zero effective width that regularizes the singularity
- the effective width depends on the temperature and the mediator's momentum (momentum transfer) and behaves in an expected, natural way

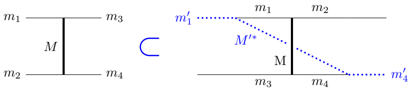
$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\mathbf{p}|) \xrightarrow{T \to 0} 0$$

- unstable initial particle necessary for the singularity to occur
  - $\Rightarrow$ no asymptotic states, what is σ?
  - $\Rightarrow$ a strict QFT approach necessary

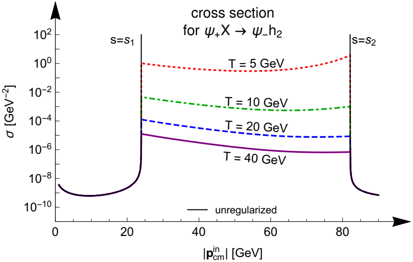
Kadanoff-Boym equations?



a larger diagram?



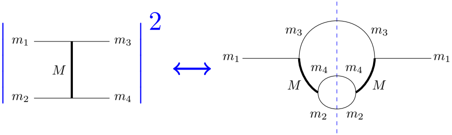
- despite the regularization, the cross section is still huge
  - $\Rightarrow$ resonant-like behaviour
  - $\Rightarrow$ impact on DM phenomenology to be carefully investigated



Summary: outlook

- unstable initial particle necessary for the singularity to occur
  - $\Rightarrow$ no asymptotic states, what is σ?
  - $\Rightarrow$ a strict QFT approach necessary

Kadanoff-Boym equations?



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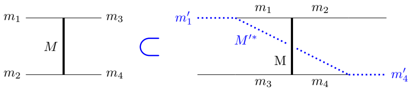

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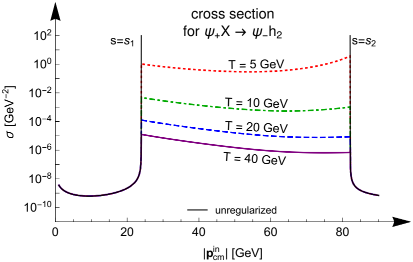


thank you!

BACKUP SLIDES

Boltzmann equation: details

E. Kolb & M. Turner, The Early Universe
P. Gondolo & G. Gelmini, Nucl.Phys.B 360 (1991) 145-179

$$\dot{n}_x + 3\,H\,n_x = -\sum_{2\to F} C^x_{ij\to f_1\dots f_F}\,\langle\sigma v\rangle_{ij\to f_1\dots f_F}\left[n_i n_j - \bar{n}_i\bar{n}_j\,\frac{n_{f_1}\dots n_{f_F}}{\bar{n}_{f_1}\dots\bar{n}_{f_F}}\right]$$

$$-\sum_{1\to F} C^x_{i\to f_1\dots f_F}\,\langle\Gamma\rangle_{i\to f_1\dots f_F}\left[n_i - \bar{n}_i\,\frac{n_{f_1}\dots n_{f_F}}{\bar{n}_{f_1}\dots\bar{n}_{f_F}}\right]$$

n, $\bar{n}$ — number density and equilibrium number density
H — Hubble parameter

combinatoric factors:

$$C^x_{ij\to f_1\dots f_F} \equiv \frac{\delta_{ix}+\delta_{jx}-\delta_{f_1x}-\dots-\delta_{f_Fx}}{1+\delta_{ij}}\,, \quad C^x_{i\to f_1\dots f_F} \equiv \delta_{ix}-\delta_{f_1x}-\dots-\delta_{f_Fx}$$

thermally averaged cross section and decay width:

$$\langle\sigma v\rangle_{ij\to f_1\dots f_F} = \frac{g_i g_j}{\bar{n}_i\bar{n}_j}\int\frac{d^3p_i}{(2\pi)^3}\frac{d^3p_j}{(2\pi)^3}\,\bar{f}_i(p_i)\bar{f}_j(p_j)\,v_{ij}\,\sigma_{ij\to f_1,f_2,\dots,f_F}$$

$$\langle\Gamma\rangle_{i\to f_1,f_2,\dots,f_F} = \frac{g_i}{\bar{n}_i}\int\frac{d^3p_i}{(2\pi)^3}\,\bar{f}_i(p_i)\,\frac{m_1}{E_1}\,\Gamma_{i\to f_1,f_2,\dots,f_F}$$

$\bar{f}(p)$ — equilibrium distribution function (Bose-Einstein or Fermi-Dirac)
g — internal degrees of freedom

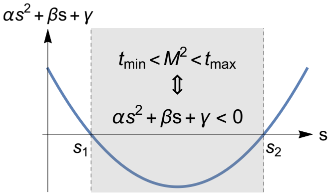
Møller velocity: $v_{ij} \equiv \sqrt{(\mathbf{v}_i-\mathbf{v}_j)^2-(\mathbf{v}_i\times\mathbf{v}_j)^2} = \dfrac{\left[(p_ip_j)^2-m_i^2m_j^2\right]^{1/2}}{E_iE_j}$

Values of s_1, s_2 in terms of masses

- in terms of the CM energy ($\sqrt{s}$):

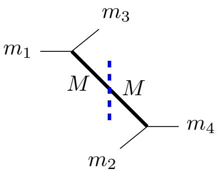
$$t_{\min}(s) < M^2 < t_{\max}(s)$$
$$\Leftrightarrow \quad s_1 < s < s_2$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$



$$\alpha \equiv M^2$$
$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)$$
$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$

Known approaches to the problem
$\longrightarrow$ complex mass of unstable particles



idea: finite lifetime should affect the wavefunction

- at rest: $$e^{im_1 t} \to e^{im_1 t} e^{-\Gamma_1 t} = e^{i\widetilde{m}_1 t}, \qquad \widetilde{m}_1 \equiv m_1\left(1 + i\frac{\Gamma_1}{m_1}\right)$$
- after Lorentz boost: $$p_1 \to \widetilde{p}_1 \equiv p_1\left(1 + i\frac{\Gamma_1}{m_1}\right)$$

$\to$ problem: $(\widetilde{p}_1 - \widetilde{p}_3)^2 \neq (\widetilde{p}_4 - \widetilde{p}_2)^2 \Rightarrow$ lack of symmetry
(momentum conservation...)

Known approaches to the problem
$\longrightarrow$ finite beam width

G. L. Kotkin et al., Yad. Fiz. 42 (1982) 692
G. L. Kotkin et al., Int. Journ. Mod. Phys. A 7 (1992) 4707
K. Melnikov & V. G. Serbo, Nucl.Phys. B483 (1997) 67
C. Dams & R. Kleiss, Eur.Phys.J.C29 (2003) 11
C. Dams & R. Kleiss, Eur.Phys.J. C36 (2004) 177

idea: at colliders, the beams have finite size

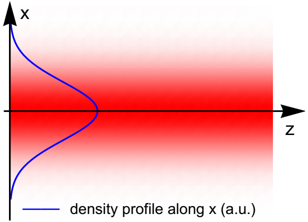
$\Downarrow$

they should not be treated as plain waves

example:

Gaussian beam moving along z axis

$$n(x, y) \sim e^{-\frac{x^2+y^2}{2a^2}} \qquad a \text{ – beam width}$$



$$\int \frac{dt}{|t - M^2 + i\epsilon|^2} \to \int \frac{a^3\, e^{-\frac{a^2\kappa^2}{2}}}{(2\pi)^{3/2}} \frac{d^3\kappa\, dt}{(t - M^2 + i\epsilon - \kappa\cdot\mathbf{q})(t - M^2 - i\epsilon + \kappa\cdot\mathbf{q})}$$

$$\sim \frac{\pi a}{|\mathbf{q}|}, \qquad \mathbf{q} \equiv \left[\frac{E_3}{E_1}\mathbf{p}_1 - \mathbf{p}_3\right]_{t=M^2}$$

$\longrightarrow$ problem: inapplicable in cosmological context

$$\underset{p}{\overset{i\Delta}{- - -}} = \underset{p}{\overset{i\Delta^{(0)}}{- - -}} + \underset{p}{\overset{i\Delta^{(0)}}{- - -}} \, i\Pi \, \underset{p}{\overset{i\Delta^{(0)}}{- - -}} + \underset{p}{\overset{i\Delta^{(0)}}{- - -}} \, i\Pi \, \underset{p}{\overset{i\Delta^{(0)}}{- - -}} \, i\Pi \, \underset{p}{\overset{i\Delta^{(0)}}{- - -}} + \ldots$$

assumptions: $|\Pi|$ small, $p^2 \simeq M^2$

scalar: $$\frac{1}{p^2 - M^2} \to \frac{1}{p^2 - M^2 + \Pi}$$

fermion: $$\frac{\not{p} + M}{p^2 - M^2} \to \frac{\not{p} + M}{p^2 - M^2 + \mathrm{Tr}\left[\frac{\not{p}+\mathrm{M}}{2}\Pi\right]}$$

vector: $$\frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}}{p^2 - M^2} \to \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}}{p^2 - M^2 + \frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\mu\nu}}$$

$$\Rightarrow \quad \text{regulator:} \quad \Sigma \equiv \begin{cases} \Im\,\Pi \\ \Im\left(\mathrm{Tr}\left[\frac{\not{p}+\mathrm{M}}{2}\Pi\right]\right) \\ \Im\left[\frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\mu\nu}\right] \end{cases}$$

→ problem:

$$\Sigma \xrightarrow[\text{opt. th.}]{p^2 \to M^2} [\text{decay width}] \Rightarrow$$ no regularization for a stable mediator

$$\overset{i\Delta}{\underset{p}{- - - -}} = \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} + \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} \, i\Pi \, \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} + \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} \, i\Pi \, \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} \, i\Pi \, \overset{i\Delta^{(0)}}{\underset{p}{- - - -}} + \dots$$

assumptions: $|\Pi|$ small, $p^2 \simeq M^2$

scalar: $$\frac{1}{p^2 - M^2} \rightarrow \frac{1}{p^2 - M^2 + \Pi}$$

fermion: $$\frac{\not{p} + M}{p^2 - M^2} \rightarrow \frac{\not{p} + M}{p^2 - M^2 + \mathrm{Tr}\left[\frac{\not{p}+\mathrm{M}}{2}\Pi\right]}$$

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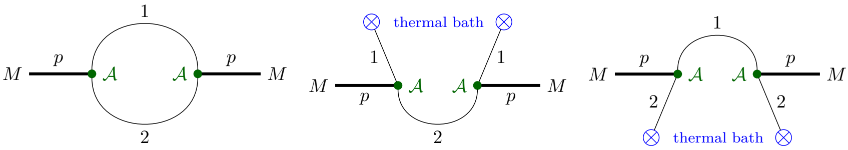
$\Rightarrow$ regulator: $$\Sigma \equiv \begin{cases} \Im\,\Pi \\ \Im\left(\mathrm{Tr}\left[\frac{\not{p}+\mathrm{M}}{2}\Pi\right]\right) \\ \Im\left[\frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\mu\nu}\right] \end{cases}$$

→ problem:

$$\Sigma \xrightarrow[\text{opt. th.}]{p^2 \to M^2} [\text{decay width}] \overset{?}{\Rightarrow} \text{no regularization for a } \underline{\text{stable}} \text{ mediator}$$

- one-loop contribution to the self-energy

$$i\Pi(x,y) = i\Delta_1(x,y)\,\mathcal{A}(y)\,i\Delta_2(y,x)\,\mathcal{A}(x)$$



- non-zero imaginary part of the self-energy appears as a result of interactions with the thermal medium of particles

$$\Pi^+(p,T) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\mathcal{A}\,\Delta_2^{\text{sym}}(k,T)\,\mathcal{A} + \Delta_1^{\text{sym}}(k,T)\,\mathcal{A}\,\Delta_2^-(k-p)\,\mathcal{A}\Big]$$

$$\Delta_i^{\text{sym}}(k,T) \equiv \frac{i\pi}{E_i}\Big(\delta(E_i-k_0)+\delta(E_i+k_0)\Big) \times \big[2\,\eta_i\,f(E_i,T)-1\big] \times (\text{numerator})$$

$$\Delta_i^{\pm}(p) \equiv \frac{(\text{numerator})}{p^2-m_i^2 \pm i\,\text{sgn}(p_0)\,\varepsilon}\,, \quad E_i \equiv \sqrt{\mathbf{k}^{\,2}+m_i^2}\,,$$

$$f(E_i,T) = (e^{E_i/T}+\eta_i)^{-1}\,, \quad \eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons}$$

Result discussion: general properties

$$\Sigma(|\mathbf{p}|,T) = \frac{1}{16\pi}\,\frac{X_0}{\beta|\mathbf{p}|}\,\ln\left[1+\eta_2\,\frac{e^{-\beta(b-a)}e^{\beta E_p}\left(1-e^{-2\beta a}\right)\left(1-\eta_1\eta_2\,e^{-\beta E_p}\right)}{\left(1+\eta_1\,e^{-\beta(b-a)}\right)\left(1+\eta_2\,e^{-\beta(b+a)}e^{\beta E_p}\right)}\right]$$

$$a \equiv \frac{\lambda(m_1^2,m_2^2,M^2)^{1/2}}{2\,M^2}\,|\mathbf{p}|\,, \qquad b \equiv \frac{m_1^2-m_2^2+M^2}{2\,M^2}\,E_p\,, \qquad E_p \equiv \sqrt{\mathbf{p}^{\,2}+M^2}$$

$$\lambda(m_1^2,m_2^2,M^2) \equiv \left[m_1^2-(m_2+M)^2\right]\left[m_1^2-(m_2-M)^2\right]$$

$$\eta_i \equiv +1 \text{ for fermions, } -1 \text{ for bosons}\,, \qquad X_0 = \eta_2\,|\mathcal{M}|^2_{\text{dec}} \times \begin{cases} 1 & \text{scalar} \\ 1/2 & \text{fermion} \\ 1/3 & \text{vector} \end{cases}$$

observations:

- $b > a + E_p$, since $E_p > |\mathbf{p}|$ and $m_1^2 - m_2^2 - M^2 > \lambda^{1/2}$
- $a > 0$ and $E_p > 0$
- a, b and E_p do not depend on T
- sgn(logarithmic part) $= \eta_2 =$ sgn(X_0) $\quad\Rightarrow\quad$ $\Sigma > 0$

Result discussion: limiting cases

$$\boxed{\Sigma(|\mathbf{p}|,T) = \frac{1}{16\pi}\,\frac{X_0}{\beta|\mathbf{p}|}\,\ln\left[1+\eta_2\,\frac{e^{-\beta(b-a)}e^{\beta E_p}\left(1-e^{-2\beta a}\right)\left(1-\eta_1\eta_2\,e^{-\beta E_p}\right)}{\left(1+\eta_1\,e^{-\beta(b-a)}\right)\left(1+\eta_2\,e^{-\beta(b+a)}e^{\beta E_p}\right)}\right]}$$

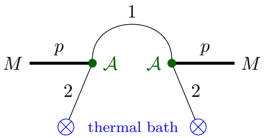
- $m_1 = m_2 + M$ (no decay)

$$a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2\,M^2}\,|\mathbf{p}| \;\to\; 0 \qquad \Rightarrow \qquad \Sigma \to 0$$

- $\beta \to \infty$ (zero temperature) or $\mathbf{p} \to \infty$

$$\beta\,a\,,\;\beta\,b\,,\;\beta\,E_p \;\to\; \infty \qquad \Rightarrow \qquad \ln[1+\ldots] \;\to\; 0 \qquad \Rightarrow \qquad \Sigma \to 0$$

  - ⋆ minimal energy E_2 needed to produce particle 1 on-shell increases with $|\mathbf{p}|$
    $\Rightarrow$ statistical suppression
  - ⋆ zero temperature $\leftrightarrow$ no medium: $f(E,T) \to 0$



- $\mathbf{p} \to 0$ (mediator at rest)

$$\Sigma \to \frac{X_0}{16\,\pi\,M}\,\frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{M^2}\,\frac{\eta_2\,e^{-\beta(b_0-M)}\left(1-\eta_1\eta_2 e^{-\beta M}\right)}{\left(1+\eta_1 e^{-\beta b_0}\right)\left(1+\eta_2 e^{-\beta(b_0-M)}\right)}$$

finite result

$$b_0 \equiv (m_1^2 - m_2^2 + M^2)/(2M) > M$$